A Martingale Approach for Fractional Brownian Motions and Related Path Dependent PDEs

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Outline

1. Introduction
2. Heat equation
3. Functional Itô formula
4. Nonlinear extension

The standard risk neutral pricing

- Let S be an underlying asset price, $\mathbb{P}$ a risk neutral measure :

$$dS_t = \sigma(t, S_t)dB_t$$

- Let $\xi = g(S_T)$ be a payoff at T, then the price at t is :

$$Y_t = \boldsymbol{E}_t[\xi]$$

- In the above Markovian setting : $Y_t = u(t, S_t)$,

$$\partial_t u + \frac{1}{2}\sigma^2(t,x)\partial^2_{xx} u = 0, \quad u(T,x) = g(x).$$

- In path dependent setting : $\sigma = \sigma(t, S_\cdot), \xi = g(S_\cdot)$, then

$$Y_t = u(t, S_\cdot),$$

$$\partial_t u + \tfrac{1}{2}\sigma^2(t,\omega)\partial^2_{\omega\omega} u = 0, \quad u(T,\omega) = g(\omega).$$

Rough volatility model

- Rough volatility : $dS_t = S_t \sigma_t dB_t$ and σ is rough
  - ◇ See e.g. Gatheral-Jaisson-Rosenbaum (2014)
- A natural model : σ driven by a fractional Brownian motion B^H
- Goal : characterize $Y_t := \boldsymbol{E}\big[\xi \mid \mathcal{F}_t^{B,B^H}\big]$
  - ◇ σ (hence B^H) can be observed
  - ◇ To focus on the main idea we will assume ξ is $\mathcal{F}_T^{B^H}$-measurable and consider $Y_t = \boldsymbol{E}\big[\xi \mid \mathcal{F}_t^{B^H}\big]$
  - ◇ Sone related recent works : El Euch-Rosenbaum (2017), Fouque-Hu (2017)

Outline

Fractional Brownian Motion

- Let B^H be a fBM with $0 < H < 1$:
  - $B^H_t - B^H_s \sim \text{Normal}(0, (t-s)^{2H})$
  - $B^H = B$ when $H = \frac{1}{2}$
- Two main features :
  - B^H is not Markovian ($H \neq \frac{1}{2}$)
  - B^H is not a semimartingale ($H < \frac{1}{2}$)
- Our goal : characterize $Y_t := \boldsymbol{E}\left[g(B^H_\cdot)\,\middle|\mathcal{F}^{B^H}_t\right]$

Heat equation in BM case

- Let $\xi := g(B_T)$ and $Y_t := \boldsymbol{E}_t[g(B_T)]$.

- Denote

$$v(t,x) := \boldsymbol{E}\Big[g(x + B_T - B_t)\Big] = \int_{\mathbb{R}} g(y)p(T-t, y-x)dy$$

$$\text{where} \quad p(t,x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

- Heat equation :

$$\partial_t p(t,x) - \tfrac{1}{2}\partial_{xx} p(t,x) = 0$$

$$\partial_t v(t,x) + \tfrac{1}{2}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_t = v(t, B_t)$, $0 \le t \le T$

A Heat equation for fBM

- Let $\xi := g(B^H_T)$ and $Y_t := \boldsymbol{E}_t[g(B^H_T)]$.

- Denote

$$v(t,x) := \boldsymbol{E}\big[g(x + B^H_T - B^H_t)\big] = \int_{\mathbb{R}} g(y) p_H(T-t, y-x) dy$$

$$\text{where} \quad p_H(t,x) := \frac{1}{\sqrt{2\pi}t^H} e^{-\frac{x^2}{2t^{2H}}}.$$

- Heat equation :

$$\partial_t v(t,x) + Ht^{2H-1}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_0 = v(0,0)$

A heat equation for fBM

- Let $\xi := g(B_T^H)$ and $Y_t := \boldsymbol{E}_t[g(B_T^H)]$.
- Denote $v(t,x) := \boldsymbol{E}\left[g(x + B_T^H - B_t^H)\right]$
- Heat equation :

$$\partial_t v(t,x) + Ht^{2H-1}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $Y_0 = v(0, B_0^H)$, $Y_T = v(T, B_T^H)$
- However, $v(t, B_t^H)$ is not a martingale :

$$Y_t \neq v(t, B_t^H) \text{ for } 0 < t < T.$$

A crucial representation of fBM

- Representation : $B_t^H = \int_0^t K(t,r)dW_r$
  - $\diamond$ $\mathbb{F} := \mathbb{F}^{B^H} = \mathbb{F}^W$
  - $\diamond$ $K(t,r) \sim (t-r)^{2H-1}$, which blows up at $t=r$ when $H<\frac{1}{2}$

- Decomposition :

$$B_T^H = \int_0^T K(T,r)dW_r = \int_0^t K(T,r)dW_r + \int_t^T K(T,r)dW_r$$

$\diamond$ $\int_0^t K(T,r)dW_r$ is $\mathcal{F}_t$-measurable

$\diamond$ $\int_t^T K(T,r)dW_r$ is independent of $\mathcal{F}_t$

$\diamond$ The previous decomposition $B_T^H = B_t^H + [B_T^H - B_t^H]$ does not satisfy this property

An alternative heat equation

- Let $\xi := g(B_T^H)$ and

$$Y_t = \boldsymbol{E}_t\Big[g\Big(\int_0^t K(T,r)dW_r + \int_t^T K(T,r)dW_r\Big)\Big]$$

- Denote $v(t,x) := \boldsymbol{E}\Big[g\big(x + \int_t^T K(T,r)dW_r\big)\Big]$

- Then $Y_t = v\big(t, \int_0^t K(T,r)dW_r\big), 0 \le t \le T$

- Note : $v\big(t, \int_0^t K(T,r)dW_r\big)$ is a martingale

- Heat equation :

$$\partial_t v(t,x) + \tfrac{1}{2}K^2(T,t)\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

A closer look

- $\Theta_T^t := \int_0^t K(T,r)dW_r = \boldsymbol{E}_t[B_T^H]$ is $\mathcal{F}_t$-measurable
  - Θ_T^t is the forward variance and is observable in market
- Three ways to express Y_t :

$$Y_t = v_1(t, B^H_{t\wedge\cdot}) = v_2(t, W_{t\wedge\cdot}) = v(t, \Theta_T^t)$$

  - B^H is not a semimartingale
  - W is a martingale (of course) but v_2 is not continuous
  - v has desired regularity and $t \mapsto \Theta_T^t$ is a martingale

An extension

- Denote $Y_t := \boldsymbol{E}_t\Big[g(B_T^H) + \int_t^T f(s, B_s^H)ds\Big]$.
- By previous computation :

$$
\begin{aligned}
Y_t &= \boldsymbol{E}_t[g(B_T^H)] + \int_t^T \boldsymbol{E}_t[f(s, B_s^H)]ds \\
&= v(T, g; t, \boldsymbol{E}_t[B_T^H]) + \int_t^T v(s, f(s,\cdot); t, \boldsymbol{E}_t[B_s^H])ds \\
&= u(t, \{\boldsymbol{E}_t[B_s^H]\}_{t\le s\le T})
\end{aligned}
$$

- Note : u is path dependent
  - ◇ If $H = \frac{1}{2}$, $\boldsymbol{E}_t[B_s] = B_t$, so $V_t = u(t, B_t)$ is state dependent
  - ◇ In more general cases,

$$
Y_t = u\Big(t, \{B_s^H\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B_s^H]\}_{t\le s\le T}\Big).
$$

Outline

The canonical setup

- Recall

$$Y_t = u\Big(t, \{B^H_s\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B^H_s]\}_{t\le s\le T}\Big).$$

- For $t \in [0,T]$, $\omega \in \mathbb{D}^0([0,t))$, and $\theta \in C^0([t,T])$, define :

$$(\omega \otimes_t \theta)_s := \omega_s \mathbf{1}_{[0,t)}(s) + \theta_s \mathbf{1}_{[t,T]}(s), \quad 0 \le s \le T.$$

- The canonical space :

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0,T], \omega \in \mathbb{D}^0([0,t)), \theta \in C^0([t,T])\Big\};$$

$$\Lambda_0 := \Big\{(t, \omega \otimes_t \theta) \in \Lambda : \omega \in C^0([0,t]), \omega_0 = 0, \theta_t = \omega_t\Big\}.$$

Continuous mapping

- Recall

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0, t)), \theta \in C^0([t, T])\Big\}.$$

- The metric :

$$\mathbf{d}((t, \omega \otimes_t \theta), (t', \omega' \otimes_{t'} \theta'))$$

$$:= \sqrt{|t - t'|} + \sup_{0 \le s \le T} |(\omega \otimes_t \theta)_s - (\omega' \otimes_{t'} \theta')_s|.$$

- $C^0(\Lambda)$: continuous mapping $u : \Lambda \to \mathbb{R}$

- $C_b^0(\Lambda)$: bounded $u \in C^0(\Lambda)$

Path derivatives

- Time derivative :

$$\partial_t u(t, \omega \otimes_t \theta) := \lim_{\delta \downarrow 0} \frac{u(t+\delta, \omega \otimes_t \theta) - u(t, \omega \otimes_t \theta)}{\delta}.$$

  ◇ $\partial_t u$ is the right time derivative !

- First order spatial derivative : Fréchet derivative with respect to θ

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[u(t, \omega \otimes_t (\theta + \varepsilon \eta)) - u(t, \omega \otimes_t \theta) \Big],$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta \in C^0([t, T])$.

Path derivatives (cont)

- Second order spatial derivative : bilinear operator on $C^0([t,T])$:

$$\langle \partial^2_{\theta\theta} u(t,\omega\otimes_t\theta), (\eta_1,\eta_2)\rangle$$

$$:= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\langle \partial_\theta u(t,\omega\otimes_t(\theta+\varepsilon\eta_1)),\eta_2\rangle - \langle \partial_\theta u(t,\omega\otimes_t\theta),\eta_2\rangle\Big].$$

for all $(t,\omega\otimes_t\theta)\in\Lambda$, $\eta_1,\eta_2\in C^0([t,T])$.

- Define the spaces $C^{1,2}(\Lambda)$ and $C^{1,2}_b(\Lambda)$ in obvious sense

Functional Ito formula : $H \geq \frac{1}{2}$

- Regular case : $K(t,t)$ is finite and thus

$$s \in [t,T] \mapsto K_s^t := K(s,t) \text{ is in } C^0([t,T]).$$

- Denote : $X_s := B_s^H,\ 0 \leq s \leq t\,; \quad \Theta_s^t := \boldsymbol{E}_t[B_s^H],\ t \leq s \leq T$

- Functional Ito formula :

$$\begin{aligned} & du(t, X \otimes_t \Theta^t) \\ = \quad & \partial_t u(\cdot)dt + \langle \partial_\theta u(\cdot), K^t \rangle dW_t + \frac{1}{2}\langle \partial^2_{\theta\theta} u(\cdot), (K^t, K^t)\rangle dt. \end{aligned}$$

◇ If $H = \frac{1}{2}$, $K = 1$, this is exactly Dupire's functional Ito formula

Functional Ito formula : $H < \frac{1}{2}$

- $K(s,t) \sim (s-t)^{H-\frac{1}{2}}$, $\partial_s K(s,t) \sim (s-t)^{H-\frac{3}{2}}$, $0 \le t < s \le T$
- For some $\alpha > \frac{1}{2} - H$, for any $(t, \omega \otimes_t \theta) \in \Lambda_0$, any $t < t_1 < t_2 \le T$, any $\eta \in C^0([t,T]$ with support in $[t_1, t_2]$,

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle \le C[t_2 - t_1]^\alpha \|\eta\|_\infty,$$

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta, \eta) \rangle \le C[t_2 - t_1]^{2\alpha} \|\eta\|^2_\infty.$$

  ◇ Roughly speaking, we want $\partial_{\theta_t} u(t, \omega \otimes_t \theta) = 0$.

- Denote $K^{t,\delta}_s := K^t_{(t+\delta)\vee s}$. Then the following limits exist :

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), K^t \rangle := \lim_{\delta \to 0} \langle \partial_\theta u(t, \omega \otimes_t \theta), K^{t,\delta} \rangle;$$

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t) \rangle := \lim_{\delta \to 0} \langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^{t,\delta}, K^{t,\delta}) \rangle.$$

- Functional Ito formula still holds

Linear path dependent PDE

- $Y_t := \boldsymbol{E}_t\Big[g(B_T^H) + \int_t^T f(s, B_s^H)ds\Big] = u(t, X \otimes_t \Theta^t)$
- $Y_t + \int_0^t f(s, B_s^H)ds$ is a martingale
- Linear PPDE :

$$\partial_t u(t, \omega \otimes_t \theta) + \frac{1}{2}\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t)\rangle + f(t, \omega_t) = 0,$$
$$u(T, \omega) = g(\omega_T).$$

- Theorem. Assume f and g are smooth, then the above PPDE has a unique classical solution u.

Outline

Nonlinear dynamics

- Forward dynamics : Volterra SDE

$$X_t = x + \int_0^t b(t; r, X_\cdot)dr + \int_0^t \sigma(t; r, X_\cdot)dW_r$$

- Backward dynamics : BSDE

$$Y_t = g(X_\cdot) + \int_t^T f(s, X_\cdot, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

$\diamond$ The backward one itself is time consistent. If we consider Volterra type of BSDEs, see a series of works by Jiongmin Yong.

- $Y_t = u(t, X \otimes_t \Theta^t)$, where

$$\Theta^t_s := x + \int_0^t b(s; r, X_\cdot)dr + \int_0^t \sigma(s; r, X_\cdot)dW_r, \ \ t \le s \le T.$$

Nonlinear PPDE

- Representation : $u(t,\omega\otimes_t\theta) := Y_t^{t,\omega\otimes_t\theta}$, where

$$X_s^{t,\omega\otimes_t\theta} = \theta_s + \int_t^s b(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dr + \int_t^s \sigma(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dW_r$$

$$Y_s^{t,\omega\otimes_t\theta} = g(\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}) - \int_s^T Z_r^{t,\omega\otimes_t\theta}dW_r + \int_s^T f(r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}, Y_r^{t,\omega\otimes_t\theta}, Z_r^{t,\omega\otimes_t\theta})dr.$$

- Semilinear PPDE : $\varphi_s^{t,\omega} := \varphi(s;t,\omega)$, $t\le s\le T$, for $\varphi = b,\sigma$,

$$\partial_t u + \frac{1}{2}\langle\partial^2_{\theta\theta}u,(\sigma^{t,\omega},\sigma^{t,\omega})\rangle + \langle\partial_\theta u, b^{t,\omega}\rangle + f\big(t,\omega,u,\langle\partial_\theta u,\sigma^{t,\omega}\rangle\big) = 0,$$

$$u(T,\omega) = g(\omega).$$

Further research

- Controlled problems (fully nonlinear PPDE)
- Viscosity solution
- Efficient numerical algorithms

Thank you very much for your attention !