A Martingale Approach for Fractional Brownian Motions and Related Path Dependent PDEs

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1 Introduction

2 Heat equation

3 Functional Itô formula

4 Nonlinear extension
The standard risk neutral pricing

• Let $S$ be an underlying asset price, $\mathbb{P}$ a risk neutral measure:
  \[ dS_t = \sigma(t, S_t)dB_t \]

• Let $\xi = g(S_T)$ be a payoff at $T$, then the price at $t$ is:
  \[ Y_t = E_t[\xi] \]

• In the above Markovian setting: $Y_t = u(t, S_t)$,
  \[ \partial_t u + \frac{1}{2} \sigma^2(t, x) \partial_{xx}^2 u = 0, \quad u(T, x) = g(x). \]

• In path dependent setting: $\sigma = \sigma(t, S. \cdot), \xi = g(S. \cdot)$, then
  \[ Y_t = u(t, S. \cdot), \]
  \[ \partial_t u + \frac{1}{2} \sigma^2(t, \omega) \partial_{\omega\omega}^2 u = 0, \quad u(T, \omega) = g(\omega). \]
Rough volatility model

- Rough volatility: \( dS_t = S_t \sigma_t dB_t \) and \( \sigma \) is rough
  - See e.g. Gatheral-Jaisson-Rosenbaum (2014)
- A natural model: \( \sigma \) driven by a fractional Brownian motion \( B^H \)
- Goal: characterize \( Y_t := E \left[ \xi \mid \mathcal{F}_t^{B,H} \right] \)
  - \( \sigma \) (hence \( B^H \)) can be observed
  - To focus on the main idea we will assume \( \xi \) is \( \mathcal{F}_T^{B,H} \)-measurable and consider \( Y_t = E \left[ \xi \mid \mathcal{F}_t^{B,H} \right] \)
Outline

1. Introduction
2. Heat equation
3. Functional Itô formula
4. Nonlinear extension
Let $B^H$ be a fBM with $0 < H < 1$:

- $B^H_t - B^H_s \sim \text{Normal}(0, (t - s)^{2H})$
- $B^H = B$ when $H = \frac{1}{2}$

Two main features:

- $B^H$ is not Markovian ($H \neq \frac{1}{2}$)
- $B^H$ is not a semimartingale ($H < \frac{1}{2}$)

Our goal: characterize $Y_t := E[g(B^H_t) \mid \mathcal{F}^B_t]$
Let \( \xi := g(B_T) \) and \( Y_t := E_t[g(B_T)] \).

Denote

\[
\nu(t, x) := E[g(x + B_T - B_t)] = \int_{\mathbb{R}} g(y)p(T - t, y - x)dy
\]

where \( p(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \).

Heat equation:

\[
\partial_t p(t, x) - \frac{1}{2} \partial_{xx} p(t, x) = 0
\]

\[
\partial_t \nu(t, x) + \frac{1}{2} \partial_{xx} \nu(t, x) = 0, \quad \nu(T, x) = g(x).
\]

\( Y_t = \nu(t, B_t), \quad 0 \leq t \leq T \)
A Heat equation for fBM

- Let $\xi := g(B^H_T)$ and $Y_t := E_t[g(B^H_T)]$.
- Denote

$$v(t, x) := E\left[g(x + B^H_T - B^H_t)\right] = \int_{\mathbb{R}} g(y)p_H(T - t, y - x)dy$$

where $p_H(t, x) := \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{x^2}{2t^{2H}}}$.

- Heat equation:

$$\partial_t v(t, x) + Ht^{2H-1}\partial_{xx} v(t, x) = 0, \quad v(T, x) = g(x).$$

- $Y_0 = v(0, 0)$
A heat equation for fBM

• Let $\xi := g(B_T^H)$ and $Y_t := E_t[g(B_T^H)]$.

• Denote $v(t, x) := E[g(x + B_T^H - B_t^H)]$

• Heat equation:

$$\partial_t v(t, x) + Ht^{2H-1}\partial_{xx} v(t, x) = 0, \quad v(T, x) = g(x).$$

• $Y_0 = v(0, B_0^H), \quad Y_T = v(T, B_T^H)$

• However, $v(t, B_t^H)$ is not a martingale:

$$Y_t \neq v(t, B_t^H) \text{ for } 0 < t < T.$$
A crucial representation of fBM

- Representation: \( B^H_t = \int_0^t K(t, r) dW_r \)
  - \( \mathcal{F} := \mathcal{F}^{B^H} = \mathcal{F}^W \)
  - \( K(t, r) \sim (t - r)^{2H-1} \), which blows up at \( t = r \) when \( H < \frac{1}{2} \)

- Decomposition:
  \[
  B^H_T = \int_0^T K(T, r) dW_r = \int_0^t K(T, r) dW_r + \int_t^T K(T, r) dW_r
  \]
  - \( \int_0^t K(T, r) dW_r \) is \( \mathcal{F}_t \)-measurable
  - \( \int_t^T K(T, r) dW_r \) is independent of \( \mathcal{F}_t \)
  - The previous decomposition \( B^H_T = B^H_t + [B^H_T - B^H_t] \) does not satisfy this property
An alternative heat equation

- Let \( \xi := g(B^H_T) \) and

\[
Y_t = E_t \left[ g \left( \int_0^t K(T, r) dW_r + \int_t^T K(T, r) dW_r \right) \right]
\]

- Denote \( v(t, x) := E \left[ g(x + \int_t^T K(T, r) dW_r) \right] \)

- Then \( Y_t = v(t, \int_0^t K(T, r) dW_r), 0 \leq t \leq T \)

- Note: \( v(t, \int_0^t K(T, r) dW_r) \) is a martingale

- Heat equation:

\[
\partial_t v(t, x) + \frac{1}{2} K^2(T, t) \partial_{xx} v(t, x) = 0, \quad v(T, x) = g(x).
\]
A closer look

- $\Theta_T^t := \int_0^t K(T, r) dW_r = E_t[B^H_T]$ is $\mathcal{F}_t$-measurable
  - $\Theta_T^t$ is the forward variance and is observable in market

- Three ways to express $Y_t$:
  - $Y_t = v_1(t, B^H_t) = v_2(t, W_t) = v(t, \Theta_T^t)$
    - $B^H$ is not a semimartingale
    - $W$ is a martingale (of course) but $v_2$ is not continuous
    - $v$ has desired regularity and $t \mapsto \Theta_T^t$ is a martingale
An extension

• Denote $Y_t := E_t \left[ g(B_H^T) + \int_t^T f(s, B_s^H) ds \right]$. 

• By previous computation:

$$Y_t = E_t[g(B_H^T)] + \int_t^T E_t[f(s, B_s^H)] ds$$

$$= v(T, g; t, E_t[B_H^T]) + \int_t^T v(s, f(s, \cdot); t, E_t[B_s^H]) ds$$

$$= u(t, \{E_t[B_s^H]\}_{t\leq s \leq T})$$

• Note: $u$ is path dependent

  ◊ If $H = \frac{1}{2}$, $E_t[B_s] = B_t$, so $V_t = u(t, B_t)$ is state dependent

  ◊ In more general cases,

$$Y_t = u\left(t, \{B_s^H\}_{0 \leq s \leq t} \otimes_t \{E_t[B_s^H]\}_{t \leq s \leq T}\right).$$
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The canonical setup

- Recall

\[ Y_t = u \left( t, \{ B_s^H \}_{0 \leq s \leq t} \otimes \{ E_t[B_s^H] \}_{t \leq s \leq T} \right). \]

- For \( t \in [0, T] \), \( \omega \in \mathbb{D}^0([0, t)) \), and \( \theta \in C^0([t, T]) \), define:

\[ (\omega \otimes_t \theta)_s := \omega_s 1_{[0, t)}(s) + \theta_s 1_{[t, T]}(s), \quad 0 \leq s \leq T. \]

- The canonical space:

\[ \Lambda := \left\{ (t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0, t)), \theta \in C^0([t, T]) \right\}; \]
\[ \Lambda_0 := \left\{ (t, \omega \otimes_t \theta) \in \Lambda : \omega \in C^0([0, t]), \omega_0 = 0, \theta_t = \omega_t \right\}. \]
Continuous mapping

• Recall

\[ \Lambda := \left\{ (t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0, t]), \theta \in C^0([t, T]) \right\}. \]

• The metric:

\[ d((t, \omega \otimes_t \theta), (t', \omega' \otimes_{t'} \theta')) := \sqrt{|t - t'|} + \sup_{0 \leq s \leq T} |(\omega \otimes_t \theta)_s - (\omega' \otimes_{t'} \theta')_s|. \]

• \( C^0(\Lambda) \): continuous mapping \( u : \Lambda \to \mathbb{R} \)

• \( C^0_b(\Lambda) \): bounded \( u \in C^0(\Lambda) \)
Path derivatives

- **Time derivative**:
  \[ \partial_t u(t, \omega \otimes_t \theta) := \lim_{\delta \downarrow 0} \frac{u(t + \delta, \omega \otimes_t \theta) - u(t, \omega \otimes_t \theta)}{\delta}. \]

  \( \partial_t u \) is the right time derivative!

- **First order spatial derivative**: Fréchet derivative with respect to \( \theta \)
  \[ \langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ u(t, \omega \otimes_t (\theta + \varepsilon \eta)) - u(t, \omega \otimes_t \theta) \right], \]

  for all \((t, \omega \otimes_t \theta) \in \Lambda, \eta \in C^0([t, T])\).
• **Second order spatial derivative**: bilinear operator on $C^0([t, T])$:

$$\langle \partial^2_{\theta \theta} u(t, \omega \otimes_t \theta), (\eta_1, \eta_2) \rangle$$

$$:= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \langle \partial_\theta u(t, \omega \otimes_t (\theta + \varepsilon \eta_1)), \eta_2 \rangle - \langle \partial_\theta u(t, \omega \otimes_t \theta), \eta_2 \rangle \right].$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta_1, \eta_2 \in C^0([t, T])$.

• Define the spaces $C^{1,2}(\Lambda)$ and $C^1_b(\Lambda)$ in obvious sense
Functional Ito formula: $H \geq \frac{1}{2}$

- **Regular case**: $K(t, t)$ is finite and thus

  $$s \in [t, T] \mapsto K^t_s := K(s, t) \text{ is in } C^0([t, T]).$$

- **Denote**: $X_s := B^H_s, 0 \leq s \leq t; \Theta^t_s := E_t[B^H_s], t \leq s \leq T$

- **Functional Ito formula**:

  $$du(t, X \otimes_t \Theta^t)$$

  $$= \partial_t u(\cdot)dt + \langle \partial_{\theta} u(\cdot), K^t \rangle dW_t + \frac{1}{2} \langle \partial^2_{\theta\theta} u(\cdot), (K^t, K^t) \rangle dt.$$

  ◯ If $H = \frac{1}{2}, K = 1$, this is exactly Dupire’s functional Ito formula
Functional Ito formula : $H < \frac{1}{2}$

- $K(s, t) \sim (s - t)^{H - \frac{1}{2}}$, $\partial_s K(s, t) \sim (s - t)^{H - \frac{3}{2}}$, $0 \leq t < s \leq T$

- For some $\alpha > \frac{1}{2} - H$, for any $(t, \omega \otimes_t \theta) \in \Lambda_0$, any $t < t_1 < t_2 \leq T$, any $\eta \in C^0([t, T]$ with support in $[t_1, t_2]$,

\[
\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle \leq C [t_2 - t_1]^\alpha \| \eta \|_\infty,
\]

\[
\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta, \eta) \rangle \leq C [t_2 - t_1]^{2\alpha} \| \eta \|_\infty^2.
\]

- Roughly speaking, we want $\partial_{\theta_t} u(t, \omega \otimes_t \theta) = 0$.

- Denote $K^{t,\delta}_s := K^{t}_{(t+\delta)\vee s}$. Then the following limits exist :

\[
\langle \partial_\theta u(t, \omega \otimes_t \theta), K^t \rangle := \lim_{\delta \to 0} \langle \partial_\theta u(t, \omega \otimes_t \theta), K^{t,\delta} \rangle;
\]

\[
\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t) \rangle := \lim_{\delta \to 0} \langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^{t,\delta}, K^{t,\delta}) \rangle.
\]

- Functional Ito formula still holds
Linear path dependent PDE

- \( Y_t := E_t \left[ g(B^H_T) + \int_s^T f(s, B^H_s) ds \right] = u(t, X \otimes t \Theta^t) \)

- \( Y_t + \int_0^t f(s, B^H_s) ds \) is a martingale

- Linear PPDE:
  \[
  \partial_t u(t, \omega \otimes_t \theta) + \frac{1}{2} \langle \partial_{\theta\theta}^2 u(t, \omega \otimes_t \theta), (K^t, K^t) \rangle + f(t, \omega_t) = 0,
  \]
  \[
  u(T, \omega) = g(\omega_T).
  \]

- **Theorem.** Assume \( f \) and \( g \) are smooth, then the above PPDE has a unique classical solution \( u \).
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Nonlinear dynamics

- Forward dynamics: Volterra SDE
  \[ X_t = x + \int_0^t b(t; r, X.) dr + \int_0^t \sigma(t; r, X.) dW_r \]

- Backward dynamics: BSDE
  \[ Y_t = g(X.) + \int_t^T f(s, X., Y_s, Z_s) ds - \int_t^T Z_s dW_s. \]

  The backward one itself is time consistent. If we consider Volterra type of BSDEs, see a series of works by Jiongmin Yong.

- \( Y_t = u(t, X \otimes_t \Theta^t) \), where
  \[ \Theta^t_s := x + \int_0^t b(s; r, X.) dr + \int_0^t \sigma(s; r, X.) dW_r, \ t \leq s \leq T. \]
• **Representation:** \( u(t, \omega \otimes_t \theta) := Y_{t}^{t, \omega \otimes t \theta} \), where

\[
X_{s}^{t, \omega \otimes t \theta} = \theta_s + \int_{t}^{s} b(s; r, \omega \otimes_t X_{.}^{t, \omega \otimes t \theta}) dr \\
+ \int_{t}^{s} \sigma(s; r, \omega \otimes_t X_{.}^{t, \omega \otimes t \theta}) dW_r \\
Y_{s}^{t, \omega \otimes t \theta} = g(\omega \otimes_t X_{.}^{t, \omega \otimes t \theta}) - \int_{s}^{T} Z_{r}^{t, \omega \otimes t \theta} dW_r \\
+ \int_{s}^{T} f(r, \omega \otimes_t X_{.}^{t, \omega \otimes t \theta}, Y_{r}^{t, \omega \otimes t \theta}, Z_{r}^{t, \omega \otimes t \theta}) dr.
\]

• **Semilinear PPDE:** \( \varphi_{s}^{t, \omega} := \varphi(s; t, \omega), \quad t \leq s \leq T \), for \( \varphi = b, \sigma \),

\[
\partial_t u + \frac{1}{2} \left< \partial_{\theta \theta}^2 u, (\sigma_{.}^{t, \omega}, \sigma_{.}^{t, \omega}) \right> + \left< \partial_\theta u, b^{t, \omega} \right> + f(t, \omega, u, \left< \partial_\theta u, \sigma^{t, \omega} \right>) = 0, \\
u(T, \omega) = g(\omega).
\]
Further research

- Controlled problems (fully nonlinear PPDE)
- Viscosity solution
- Efficient numerical algorithms
Thank you very much for your attention!