Optimal Investment, Indifference Pricing and Dynamic Default Insurance in the Presence of Defaults

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Research Goals

Solve the optimal investment problem when the underlying traded asset may default.

- Price defaultable bonds.
- Price dynamic default insurance.

Obtain explicit answers.

- Provide a PDE counterpart to the BSDE pricing literature.
Motivation

Say our goal is to price a claim whose payoff is contingent upon survival of a reference entity.

- Payoff: $\phi_T 1_{T>\delta}$
- $\delta$: default time of a firm $S$.

In practice, pricing is done under a risk neutral measure.

Two problems:

- What risk neutral measure?
- What is the underlying traded asset? What if the underlying is the reference entity?
Motivation

Say our goal is to insure ourselves against losses from the default of a stock in which we own a position.

We could enter into a CDS

- What if investment horizon does not match CDS maturity?
- What if we want dynamic protection?

Is there a fair price for dynamic protection taking into account market incompleteness, and our preferences?
Contribution to the Literature

Optimal investment and indifference pricing with defaults have been extensively studied.

- Primarily from the "BSDE" perspective, especially with respect to pricing.
- We fill in a gap by considering Markovian factor models, using PDE techniques, and focusing on indifference pricing.
  - Amenable to computation and analysis.

The computation of dynamic default insurance has been much less well-studied.
Contribution to the Literature

(selected) "PDE" articles

- [Lin06]: Merton model with default intensity $\gamma_t = \gamma(S_t)$ under a fixed risk neutral measure. Analytical formulas for European option prices.

- [SZ07]: single stock factor model similar to ours. However, investor does not lose money in stock upon default.

- [BBC16]: risk-sensitive control problem in factor model with multiple securities, default state dependent intensities. Investor does not lose money in stock upon default.

- [BC16]: optimal investment/consumption problem for power utility in a factor model with multiple securities, default state dependent intensities. Investor loses money upon default.
Contribution to the Literature

(selected) "BSDE" articles

- [Mor09, LQ11]: single stock and non-traded claim. Brownian setting prior to default.

- [JP11, JKP13]: single/multiple stocks along with claim. Multiple credit events which cause a jump in stock prices with trading possible after jump. Brownian setting

- [MS17]: stock modeled as a pure-jump Levy process.

- [LQ15]: extension of [LQ11] to partial information models.

- [GN15, CGN15]: mean-variance hedging under default risk.
Model

Reduced form, "hybrid" intensity model: [SZ07].

$X$: underlying factor process

- $dX_t = b(X_t)dt + a(X_t)dW_t.$
  - $W$: $d$-dim B.M.. $b, A := aa' \text{ smooth, } A \text{ locally elliptic.}$
  - Solution to Martingale problem for $L$ on $E \subseteq \mathbb{R}^d$ where
    - $L = (1/2)\text{Tr}(AD^2) + b'\nabla$
    - $E = \bigcup_n E_n$ with $E_n$ bounded, $E_n \uparrow$, $\partial E_n$ smooth.

One risky asset $S$ (riskless asset set to 1)

- $S$ defaults at the random time $\delta$. Prior to $\delta$, $S$ has instantaneous returns, variances, correlations driven by $X$. 
Model

Start at $t \geq 0$. $X^{t,x}_t = x \in E$. Write $X = X^{t,x}$.

\[
\frac{dS_s}{S_s} = 1_{s \leq \delta} \left( (\mu - \gamma)(X_s)ds + (\sigma \rho)(X_s)'dW_s + \left( \sigma \sqrt{1 - \rho' \rho} \right)(X_s)dW^0_s \right)
- dM_s; \quad s \geq t.
\]

- $W^0$: one-dim B.M. \(\perp\) of $W$.
- $\delta := \inf \{ s > t : \int_t^s \gamma(X_u)du = -\log(U) \}$, $U \perp W, W^0$.
- $H_s := 1_{s \geq \delta}; \quad M_s := H_s - \int_t^{s \wedge \delta} \gamma(X_u)du,$
- $G := \mathbb{F}^{W,W^0} \vee \mathbb{F}^H$. $W, W^0, M$ are $G$ local martingales.
- $\mu, \sigma, \gamma, \rho$ smooth functions on $E$, $\gamma, \sigma > 0$, $\rho' \rho \leq 1$. 
Optimal Investment Problem

Investment horizon: \([t, T]\) for \(T > t\).

\(\mathcal{M}\): equivalent local martingale measures on \(\mathcal{G}_T\). \(\tilde{\mathcal{M}}\) subset with finite relative entropy w.r.t. \(\mathbb{P}\).

\(\mathcal{A}\): acceptable (dollar) trading strategies \(\pi\).

- Wealth process \(\mathcal{W}^{\pi, w} = w + \int_t^T \pi_u dS_u / S_u\).
- Dollar position \(\pi_\delta\) lost at \(\delta\).

\(\pi \in \mathcal{A}\) if \(\mathcal{W}^{\pi, w}\) is a \(\mathbb{Q}\) local martingale for all \(\mathbb{Q} \in \tilde{\mathcal{M}}\).
Optimal Investment Problem

Exponential investor: \( U(w) := -e^{-\alpha w}, \ w \in \mathbb{R}. \)

Investor

- Trades in \( S \) according to \( \pi \in \mathcal{A} \).
- Owns a non-traded claim with time \( T \) payoff \( \phi(X_T)1_{T<\delta}. \)
  - \( \phi \) smooth, bounded. Primarily care about \( \phi \equiv 1, \phi \equiv 0. \)

For 0 initial wealth write \( \mathcal{W}^{\pi} = \mathcal{W}^{\pi,0} \) and define

\[
  u(t, x; \phi) := \sup_{\pi \in \mathcal{A}} E \left[ -e^{-\alpha (\mathcal{W}^{\pi}_t + \phi(X_T)1_{T<\delta})} \right]; \quad (X_t = x) \\
  G(t, x; \phi) := -\frac{1}{\alpha} \log (-u(t, x; \phi)).
\]
\[ G(t, x; \phi) = -\frac{1}{\alpha} \log \left( -u(t, x; \phi) \right): \text{Certainty Equivalent} \]

Heuristics using DPP suggest \( G \) should solve

\[
0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right); \quad \phi = G(T, \cdot)
\]

\[
\cdot \theta(y): \text{inverse of } ye^y \text{ and } \theta_G := \theta \left( \frac{\gamma}{\sigma^2} e^{\frac{\mu}{\sigma^2} + \alpha G} - \frac{\alpha}{\sigma} \nabla G' a \rho \right).
\]

If \( G \) is a classical solution, DPP suggests optimal strategy is

\[
\cdot \hat{\pi}_s = \hat{\pi}(s, X_s^{t,x}) \text{ for } \hat{\pi} = \frac{1}{\alpha} \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \theta_G \right).
\]
Certainty Equivalent PDE

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right) ; \]
\[ \phi = G(T, \cdot) \]

- This is a semi-linear degenerate parabolic PDE.
  - Non-linearities arise due to market incompleteness.
- Luckily: \( \theta(y) \approx \log(y) - \log(\log(y)) \), \( y \gg 0 \).
  - PDE is quadratically growing in \( G, \nabla G \).
- Regarding solutions/verification:
  - For general regions \( E \), local ellipticity, verification is hard: lack gradient estimates near \( \partial E \).
  - We must enforce some additional (global) condition.
The Main Assumption

Set \( \ell := (\mu - \gamma)/\sigma \) (market price of risk).

Today: assume "strictly incomplete" market absent default.

- The paper treats the "complete" case as well.

Main assumptions:

- \( \sup_{x \in E} \rho'(x) < 1. \)
- For some \( \varepsilon > 0 \) we have for each \( n \)

\[
\sup_{x \in E_n} \mathbb{E}^x \left[ e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(\varepsilon, n) < \infty.
\]

This assumption is MILD. Holds in virtually all models.

- E.g. \( X \sim OU, CIR, \mu, \sigma^2, \gamma \) affine.
The Main Result

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a\rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right); \]
\[ \phi = G(T, \cdot) \]

**Theorem:** assume \( \sup_{x \in E} \rho'(x) < 1 \) and for some \( \varepsilon > 0 \):
- \( \sup_{x \in \bar{E}_n} E^x \left[ e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(n) < \infty, \forall n. \)

Then
- The certainty equivalent \( G \) is a classical \((C^1,2)\) solution.
- The optimal trading strategy is
  - \( \hat{\pi}_s = \hat{\pi}(s, X_{s_t}^t) \) for \( \hat{\pi} = \frac{1}{\alpha} \left( \frac{\mu}{\sigma} - \frac{\alpha}{\sigma} \nabla G' a\rho - \theta_G \right). \)
- The optimal martingale measure \( \hat{Q} \) has density
  - \( \hat{Z}_s = e^{-\alpha \left( W_{s}^{\hat{\pi}} - G(t,x;\phi) + 1_{\delta > s} G(x,X_s,\phi) \right)} \).
Application: Pricing for Defaultable Bonds

Investor owns $q$ units notional: claim payoff $q1_{\delta \geq T}$.

(per-unit, buyer’s) indifference price: $p(t, x; q)$ solving

\[ u(t, x; 0, 0) = u(t, x; q, -qp(t, x; q)) = e^{\alpha qp(t, x; q)}u(t, x; q, 0). \]

- $u(t, x; \phi, w)$: utility for initial wealth $w$.
- Well known $p$ does not depend on $w$.

Immediate result as $G(t, x; q) = -\left(1/\alpha\right) \log(-u(t, x; q))$:

- $p(t, x; q) = \frac{1}{q} (G(t, x; q) - G(t, x; 0))$. 
Application: Dynamic Default Insurance

Goal: find a fair price for dynamic protection against default.

- Approximation to CDS pricing valid for frequent contract adjustments.

Motivation from [SZ07]: optimal investment/pricing but with no loss at default.

- $\pi_\delta$ not lost at default time $\delta$.

How is this possible? What contact has been entered into which enables this?
Dynamic Default Insurance

Perspective: investor has two alternatives:

\begin{itemize}
\item A) Do not purchase protection. Lose $\pi_{\delta}$ at $\delta$. Indirect utility of $u(t, x)$.
\item B) Purchase protection. Pay a (per-unit) cash flow rate of $f$, where $f$ is to-be-determined.
\end{itemize}

- Wealth dynamics:

$$dW_{s}^{\pi,d} = \pi_{s} \mathbf{1}_{s \leq \delta} \left( (\mu - \gamma)(X_{s}) - f_{s} \right) ds$$

$$+ \pi_{s} \mathbf{1}_{s \leq \delta} \left( (\sigma \rho)(X_{s})' dW_{s} + (\sigma \sqrt{1 - \rho' \rho})(X_{s}) dW_{s}^{0} \right).$$

- Indirect utility

$$u^{d}(t, x) := \sup_{\pi \in \mathcal{A}_{d}} E \left[ -e^{-\alpha W_{T}^{\pi,d}} \right].$$
Dynamic Default Insurance

\[ G(t, x) = -\frac{1}{\alpha} \log (-u(t, x)); \quad G^d(t, x) = -\frac{1}{\alpha} \log (-u^d(t, x)). \]

Guess \( f_t = f(t, X_t) \). Find \( f \) so that PDEs for \( G, G^d \) are the same (both have terminal condition \( \phi \)).

\[ 0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G \]
\[ + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a\rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta^2_G - 2\theta_G \right); \]

\[ 0 = G^d_t + LG^d - \frac{\alpha}{2} \nabla (G^d)' A \nabla G^d \]
\[ + \frac{\sigma^2}{2\alpha} \left( \left( \frac{\mu - f}{\sigma^2} - \frac{\alpha}{\sigma} \nabla (G^d)' a\rho \right)^2 + \frac{2\gamma}{\sigma^2} \left( 1 - e^{\alpha G^d} \right) \right). \]
Dynamic Default Insurance

Upon inspection, given a solution $G$ to the first PDE, $G$ will solve the second PDE if $f$ satisfies

$$
\frac{f_\pm}{\sigma^2} = \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho\right)^2 - \left(\theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G}\right)}.
$$

- Term inside square root is non-negative: real solutions.
- We choose the "II" solution.
  - Lowest possible $f$ since this is what the investor pays.
  - Can also justify $f_-$ by inspecting optimal strategies $\pi^d_\pm$: $f_+ > 0$ and $\pi^d_+ < 0$ - not feasible.
Dynamic Default Insurance

We define the dynamic default insurance protection price

\[ f := \sigma^2 \left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' \rho \right) - \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' \rho \right)^2 - \left( \theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G} \right)} . \]

Facts

- \[ f \leq \gamma e^{\alpha (G + \hat{\pi})} = \gamma \hat{Q} : \text{the default intensity under the dual optimal measure } \hat{Q}. \]
  - Equality only when \( \hat{\pi} = 0. \)
- \( f > 0 \text{ when } \hat{\pi} > 0: \text{intuitive. Pay for protection when long.} \)
- \( f > 0 \text{ possible even when } \hat{\pi} < 0, \text{ but } f < 0 \text{ for } \hat{\pi} << 0. \)
Numerical Application

Application: $X \sim CIR$, affine market price of risk.

- $dX_t = \kappa(\theta - X_t)dt + \xi \sqrt{X_t} dW_t$.
- Prior to default
  - $dS_t/S_t = \mu X_t dt + \sigma \sqrt{X_t} \left( \rho dW_t + \sqrt{1 - \rho^2} dW_t^0 \right)$.
- Default intensity: $\gamma_t = \gamma X_t$.

Assume $\mu \in \mathbb{R}$, $\sigma, \gamma > 0$ and $|\rho| < 1$.

- Main assumption holds provided $\kappa \theta > \xi^2/2$. 
Application: Defaultable Bond Pricing

Investor owns $q$ units of a defaultable bond.

$p(0, x; q)$ as a function of $q, x$ for $T = 1$.

- Physical default prob of 3% at $x = 6\%$ (long run mean).
- $q = 1$ (dash), $q = 3$ (dot-dash), $q = 5$ (dot), $q = 10$ (solid).
Application: Dynamic Default Insurance

$f(0, x)$ as a function of $x$ for $T = 1$.

- $\gamma^Q$ (dash), $f$ (solid), $\gamma$ (dash).
THANK YOU!
John R Birge, Lijun Bo, and Agostino Capponi, *Risk sensitive asset management and cascading defaults.*

Lijun Bo and Agostino Capponi, *Portfolio choice with market-credit risk dependecies.*


