Uncertain Volatility Models with Stochastic Bounds

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Joint work with Jean-Pierre Fouque

Uncertain Volatility Model (UVM)

- T. J. Lyons, 1995
  Uncertain volatility and the risk-free synthesis of derivatives.
- M. Avellaneda, A. Levy, and A. Paras, 1995.
  Pricing and hedging derivative securities in markets with uncertain volatilities.
- L. Denis and C. Martini, 2006.
  A theoretical framework for the pricing of contingent claims in the presence of model uncertainty.
- S. Peng, 2007.
  G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty.
- H. M. Soner, N. Touzi, and J. Zhang, 2013.
  Dual formulation of second order target problems.
- J.-P. Fouque and B. Ren, 2014.
  Approximation for option prices under uncertain volatility.

Uncertain Volatility Model (UVM)

Under the risk-neutral measure, the price process of the risky asset satisfies the following stochastic differential equation (SDE):

$$dX_t = rX_t dt + \alpha_t X_t dW_t,$$

where r is the constant risk-free rate, (W_t) is a Brownian motion and the volatility process (α_t) belongs to a family $\mathcal{A}$ of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$-valued processes.

Worst-case Scenario Price

"Uncertainty is the only certainty there is, and knowing how to live with insecurity is the only security". —John Paulos

When pricing a European derivative written on the risky asset with maturity T and nonnegative payoff $h(X_T)$, the worst-case scenario price at time $t < T$ is given by

$$P(t, X_t) := \exp(-r(T-t)) \operatorname{ess\,sup}_{\alpha \in \mathcal{A}} \mathbb{E}_t[h(X_T)].$$

Previous result

$P(t, X_t)$ solves the following Black-Scholes-Barenblatt equation:

$$\partial_t P + r(x\partial_x P^\epsilon - P) + \sup_{\alpha \in [\underline{\sigma}, \overline{\sigma}]} \{\frac{1}{2}\alpha^2 x^2 \partial^2_{xx} P\} = 0,$$
$$P(T) = h.$$

- For convex h, $P(t, X_t)$ is its Black–Scholes price with constant volatility $\overline{\sigma}$.
- For concave h, $P(t, X_t)$ is its Black–Scholes price with constant volatility $\underline{\sigma}$.
- What if h is not strictly convex nor strictly concave?

Previous result (Fouque-Ren SIFIN 2014)

For the general terminal payoff function, in a small volatility interval $[\sigma, \sigma+\epsilon]$, the worst-case scenario price $P^\epsilon(t, X_t)$ solves the following Black-Scholes-Barenblatt equation:

$$\partial_t P^\epsilon + r(x\partial_x P^\epsilon - P^\epsilon) + \sup_{\alpha\in[\sigma,\sigma+\epsilon]} \{\frac{1}{2}\alpha^2 x^2 \partial^2_{xx} P^\epsilon\} = 0,$$

$$P^\epsilon(T) = h.$$

Main result:

$$\lim_{\epsilon\downarrow 0} \frac{P^\epsilon - (P_0 + \epsilon P_1)}{\epsilon} = 0.$$

Previous result (Fouque-Ren SIFIN 2014)

P_0 is the solution of the following Black-Scholes equation:

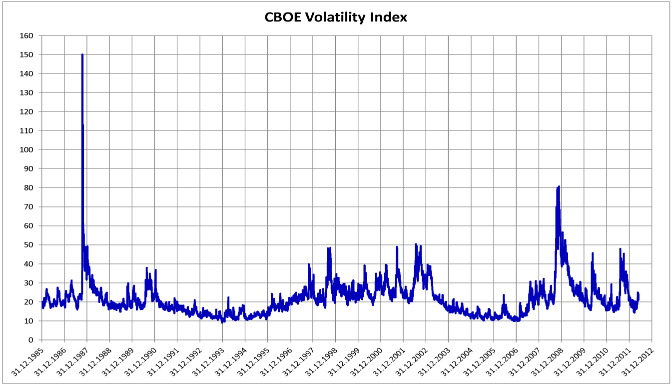
$$\begin{aligned} \partial_t P_0 + r(x\partial_x P_0 - P_0) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 P_0 &= 0, \\ P_0(T) &= h. \end{aligned}$$

P_1 is the solution of the following equation:

$$\begin{aligned} \partial_t P_1 + r(x\partial_x P_1 - P_1) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 P_1 + \sup_{g\in[0,1]} g\sigma x^2 \partial_{xx}^2 P_0 &= 0, \\ P_1(T) &= 0. \end{aligned}$$

Problem and Difficulty

However, for contingent claims with longer maturities, it is no longer consistent with observed volatility to assume that the bounds are constant.



Our Research

We propose that the uncertain volatility moves between two stochastic bounds,

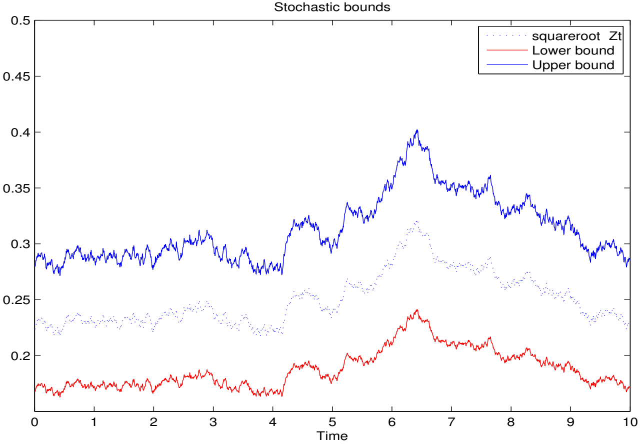
$$\underline{\sigma}_t := d\sqrt{Z_t} \leq \alpha_t \leq \overline{\sigma}_t := u\sqrt{Z_t},$$

where u and d are two constants such that $0 < d < 1 < u$, and Z_t is the general three-parameter CIR process

$$dZ_t = \delta\kappa(\theta - Z_t)dt + \sqrt{\delta}\sqrt{Z_t}dW_t^Z.$$

Denote $\alpha_t := q_t\sqrt{Z_t}$, then

$$d \leq q_t \leq u.$$


Stochastic bounds
squareroot Zt
Lower bound
Upper bound
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0
1
2
3
4
5
6
7
8
9
10
Time

Convergence of X^δ

Reparameterize the SDE of the risky asset price process as

$$dX_t^\delta = rX_t^\delta dt + q_t\sqrt{Z_t}X_t^\delta dW_t.$$

When $\delta = 0$, note that the CIR process Z_t is frozen at z, and then the risky asset price process follows the dynamic

$$dX_t^0 = rX_t^0 dt + q_t\sqrt{z}X_t^0 dW_t.$$

Both X_t^δ and X_t^0 start at the same point x.

Proposition

Uniformly in $(q_\cdot)$, $\mathbb{E}_{(t,x,z)}(X_T^\delta - X_T^0)^2 \leq C_0\delta$

Convergence of P^δ

We denote its smallest riskless selling price (worst case scenario) as

$$P^\delta(t,x,z) := \exp(-r(T-t))\operatorname{ess\,sup}_{q_\cdot\in[d,u]}\mathbb{E}_{(t,x,z)}[h(X_T^\delta)].$$

When $\delta = 0$,

$$P_0(t,x,z) = \exp(-r(T-t))\operatorname{ess\,sup}_{q_\cdot\in[d,u]}\mathbb{E}_{(t,x,z)}[h(X_T^0)].$$

Notice that $P_0(t,X_t,z)$ corresponds to $P(t,X_t)$ with constant volatility bounds given by $d\sqrt{z}$ and $u\sqrt{z}$.

Theorem

1. $P^\delta(t,\cdot,\cdot)$ as a family of functions of x and z indexed by δ, uniformly converge to $P_0(t,\cdot,\cdot)$ in $(q_\cdot)$ with rate $\sqrt{\delta}$.
2. $\partial^2_{xx}P^\delta(t,\cdot,\cdot)$ ...

Pricing Nonlinear PDE

The generalized BSB nonlinear equation:

$$\partial_t P^\delta + r(x\partial_x P^\delta - P^\delta) + \sup_{q\in[d,u]} \{\frac{1}{2}q^2 z x^2 \partial^2_{xx} P^\delta + \sqrt{\delta}(q\rho z x \partial^2_{xz} P^\delta)\}$$
$$+\delta(\frac{1}{2}z\partial^2_{zz}P^\delta + \kappa(\theta - z)\partial_z P^\delta) = 0,$$
$$P^\delta(T, x, z) = h(x).$$

Expansion:

$$P^\delta = P_0 + \sqrt{\delta}P_1 + \delta P_2 + \cdots$$

- Identify P_0 and P_1
- Control the error term $E^\delta(t,x,z) := P^\delta(t,x,z) - P_0(t,x,z) - \sqrt{\delta}P_1(t,x,z)$

Identify P_0

Inserting this expansion into the main BSB equation, the leading order term P_0 is the solution to

$$\partial_t P_0 + \sup_{q\in[d,u]}\{\frac{1}{2}q^2 z x^2 \partial_{xx}^2 P_0\} = 0,$$
$$P_0(T,x,z) = h(x).$$

The optimal control for P_0 is given by

$$q^{*,0}(t,x,z) = \begin{cases} u, \partial_{xx}^2 P_0 \geq 0 \\ d, \partial_{xx}^2 P_0 < 0 \end{cases}.$$

Optimizers

Lemma

For δ sufficiently small and for $x \notin S^0_{t,z}$ (the zero set of $\partial^2_{xx} P_0$), the optimal control in the nonlinear PDE for P^δ, denoted as

$$q^{*,\delta}(t,x,z) := \arg \max_{q \in [d,u]} \{ \frac{1}{2} q^2 z x^2 \partial^2_{xx} P^\delta + \sqrt{\delta} (q \rho z x \partial^2_{xz} P^\delta) \},$$

is given by

$$q^{*,\delta}(t,x,z) = \begin{cases} u, \partial^2_{xx} P^\delta \geq 0 \\ d, \partial^2_{xx} P^\delta < 0 \end{cases}.$$

Optimizers

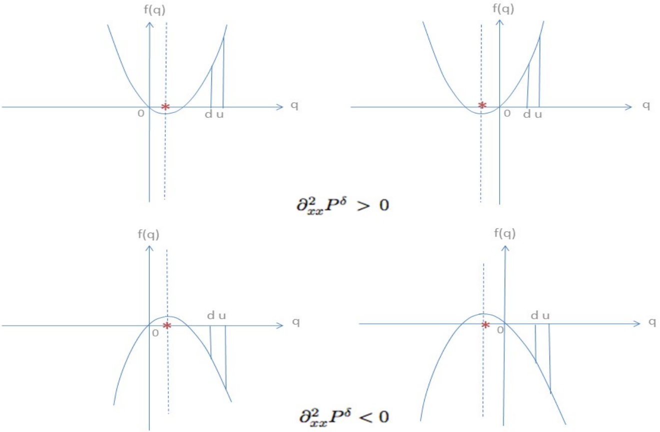


Figure: *If $\partial^2_{xx}P^\delta > 0$, whether $\hat{q}^{*,\delta}$ is positive or negative, with the requirement $q \in [d, u]$, $q^{*,\delta} = u$; otherwise $q^{*,\delta} = d$.*

Identify P_1

We insert the expansion into the main BSB equation and collect terms in successive powers of $\sqrt{\delta}$. With the result that $q^{*,\delta} \to q^{*,0}$ as $\delta \to 0$, the first order correction term P_1 is chosen as the solution to the linear equation:

$$\partial_t P_1 + \frac{1}{2}(q^{*,0})^2 z x^2 \partial_{xx}^2 P_1 + q^{*,0} \rho z x \partial_{xz}^2 P_0 = 0,$$
$$P_1(T, x, z) = 0,$$

where $q^{*,0}$ is the optimal control for P_0.

Main Theorem

Theorem (Main Theorem)

The residual function $E^\delta(t,x,z)$ defined by

$$E^\delta(t,x,z) := P^\delta(t,x,z) - P_0(t,x,z) - \sqrt{\delta}P_1(t,x,z)$$

is of order $\mathcal{O}(\delta)$. In other words, $\forall(t,x,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}^+$, there exists a positive constant C, such that $|E^\delta(t,x,z)| \leq C\delta$, where C may depend on (t,x,z) but not on δ.

Operators

$$
\begin{aligned}
\mathcal{L}^\delta(q) :&= \partial_t + \frac{1}{2}q^2 zx^2\partial^2_{xx} + \sqrt{\delta}q\rho zx\partial^2_{xz} + \delta(\frac{1}{2}z\partial^2_{zz} + \kappa(\theta - z)\partial_z) \\
&= \mathcal{L}_0(q) + \sqrt{\delta}\mathcal{L}_1(q) + \delta\mathcal{L}_2,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{L}_0(q) &:= \partial_t + \frac{1}{2}q^2 zx^2\partial^2_{xx}, \\
\mathcal{L}_1(q) &:= q\rho zx\partial^2_{xz}, \\
\mathcal{L}_2 &:= \frac{1}{2}z\partial^2_{zz} + \kappa(\theta - z)\partial_z.
\end{aligned}
$$

Apply $\mathcal{L}^\delta(q^{*,\delta})$ to the error term

$$
\begin{aligned}
& \mathcal{L}^\delta(q^{*,\delta})E^\delta \\
=& \mathcal{L}^\delta(q^{*,\delta})(P^\delta - P_0 - \sqrt{\delta}P_1) \\
=& -\underbrace{\mathcal{L}_{BS}(q^{*,0})P_0}_{=0,\ \text{PDE for } P_0.} - (\mathcal{L}_{BS}(q^{*,\delta}) - \mathcal{L}_{BS}(q^{*,0}))P_0 \\
& - \sqrt{\delta}\Bigg[\underbrace{\mathcal{L}_1(q^{*,0})P_0 + \mathcal{L}_{BS}(q^{*,0})P_1}_{=0,\ \text{PDE for } P_1.} \\
& + (\mathcal{L}_1(q^{*,\delta}) - \mathcal{L}_1(q^{*,0}))P_0 + (\mathcal{L}_{BS}(q^{*,\delta}) - \mathcal{L}_{BS}(q^{*,0}))P_1\Bigg] \\
& - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right] - \delta^{\frac{3}{2}}(\mathcal{L}_{CIR}P_1)
\end{aligned}
$$

Probabilistic Representation by Feynman-Kac

Given terminal condition

$$E^{\delta}(T,x,z)=0$$

together with the existence and uniqueness result of $X_t^{*,\delta}$, we have the following probabilistic representation for $E^{\delta}(t,x,z)$ by Feynman-Kac formula:

$$E^{\delta}(t,x,z)=I_0+\delta^{\frac{1}{2}}I_1+\delta I_2+\delta^{\frac{3}{2}}I_3,$$

$$
\begin{aligned}
I_0 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right)Z_s(X_s^{*,\delta})^2\partial^2_{xx}P_0(s,X_s^{*,\delta},Z_s)ds\right],\\
I_1 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \rho(q^{*,\delta}-q^{*,0})Z_sX_s^{*,\delta}\partial^2_{xz}P_0(s,X_s^{*,\delta},Z_s)\right.\\
&\qquad\left.+\frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right)Z_s(X_s^{*,\delta})^2\partial^2_{xx}P_1(s,X_s^{*,\delta},Z_s)ds\right],\\
I_2 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \rho(q^{*,\delta})Z_sX_s^{*,\delta}\partial^2_{xz}P_1(s,X_s^{*,\delta},Z_s)\right.\\
&\qquad\left.+\frac{1}{2}Z_s\partial^2_{zz}P_0(s,X_s^{*,\delta},Z_s)+\kappa(\theta-Z_s)\partial_zP_0(s,X_s^{*,\delta},Z_s)ds\right],\\
I_3 &:= \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}Z_s\partial^2_{zz}P_1(s,X_s^{*,\delta},Z_s)+\kappa(\theta-Z_s)\partial_zP_1(s,X_s^{*,\delta},Z_s)ds\right]
\end{aligned}
$$

Next observe that,

$$q^{*,\delta} - q^{*,0} = (u-d)(\mathbb{1}_{\{\partial^2_{xx}P^\delta \geq 0\}} - \mathbb{1}_{\{\partial^2_{xx}P_0 \geq 0\}}).$$

and similarly,

$$(q^{*,\delta})^2 - (q^{*,0})^2 = (u^2-d^2)(\mathbb{1}_{\{\partial^2_{xx}P^\delta \geq 0\}} - \mathbb{1}_{\{\partial^2_{xx}P_0 \geq 0\}}).$$

Note that $\{q^{*,\delta} \neq q^{*,0}\} = A^\delta_{t,z}$, the set where $\partial^2_{xx}P^\delta$ and $\partial^2_{xx}P_0$ take different signs

$$\begin{aligned} A^\delta_{t,z} :=& \{x = x(t,z) | \partial^2_{xx}P^\delta(t,x,z) > 0, \partial^2_{xx}P_0(t,x,z) < 0\} \\ & \cup \{x = x(t,z) | \partial^2_{xx}P^\delta(t,x,z) < 0, \partial^2_{xx}P_0(t,x,z) > 0\}. \end{aligned}$$

Control of the term I_0

Theorem

There exists a positive constant M_0, such that

$$|I_0| \leq M_0 \delta$$

where M_0 may depend on (t, x, z) but not on δ. That is, I_0 is of order $\mathcal{O}(\delta)$.

Sketch of Proof: There exists a constant C_0 such that

$$|\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)| \leq C_0 \sqrt{\delta}, \text{ for } X_s^{*,\delta} \in A_{s,z}^{\delta}.$$

Control of the term I_0

Then, since $0 < d \leq q^{*,\delta}, q^{*,0} \leq u$, we have

$$|I_0| \leq \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}|(q^{*,\delta})^2 - (q^{*,0})^2|Z_s(X_s^{*,\delta})^2|\partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s)|ds\right]$$
$$\leq \frac{u^2}{2d^2}C_0\sqrt{\delta}\,\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,z}^{\delta}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2 ds\right].$$

In order to show that I_0 is of order $\mathcal{O}(\delta)$, it suffices to show that there exists a constant C_1 such that

$$\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,z}^{\delta}\}}\sigma_s^2 ds\right] \leq C_1\sqrt{\delta},$$

where $\sigma_s := q^{*,\delta}\sqrt{Z_s}X_s^{*,\delta}$ and $dX_s^{*,\delta} = \sigma_s dW_s$.

Control of the term I_0

Define the stopping time

$$\tau(v) := \inf\{s > t; \langle X^{*,\delta}\rangle_s > v\},$$

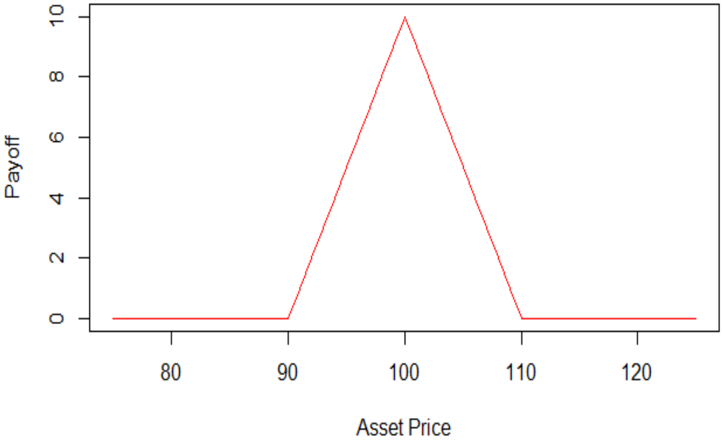
where

$$\langle X^{*,\delta}\rangle_s = \int_t^s \sigma^2(X_u^{*,\delta})du.$$

Then according to Theorem of time-change for martingales, we know that $X^{*,\delta}_{\tau(v)} = B_v$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}_v^B, \mathbb{Q}^B)$.

$$\int_t^{\tau(v)} \sigma^2(X_s^{*,\delta})ds = v, \qquad \tau^{-1}(T) = \int_t^T \sigma^2(X_s^{*,\delta})ds.$$

Symmetric European butterfly spread



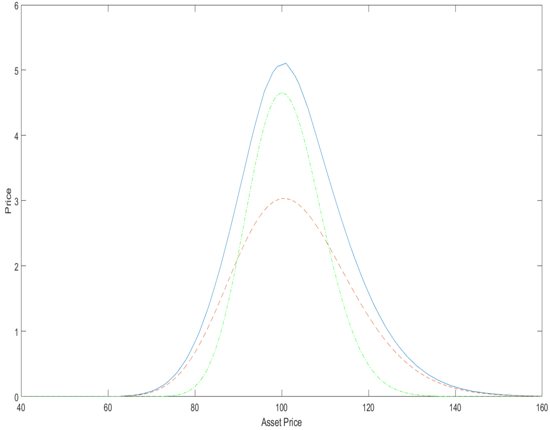
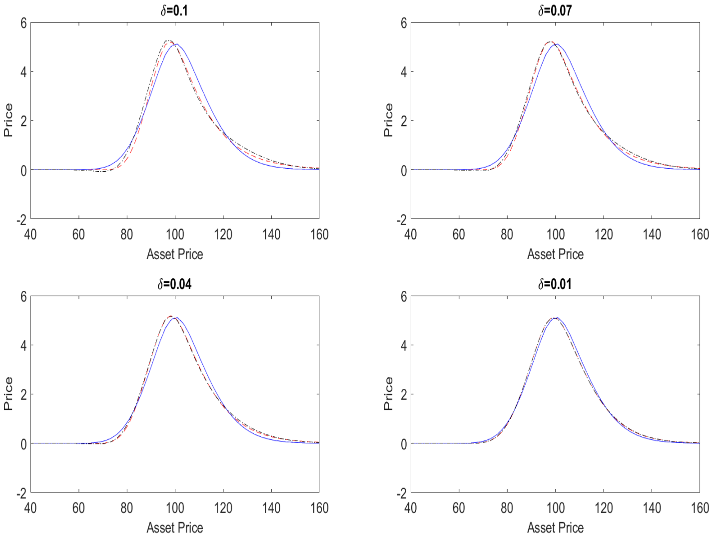


Figure: *The blue curve represents the usual uncertain volatility model price P_0 with two deterministic bounds* 0.15 *and* 0.25*, the red curve marked with "- -" represents the BS prices with $\sigma = 0.25$, the green curve marked with '-.' represents the BS prices with $\sigma = 0.15$.*


δ=0.1
δ=0.07
δ=0.04
δ=0.01
Price
Asset Price
6
4
2
0
-2
40
60
80
100
120
140
160

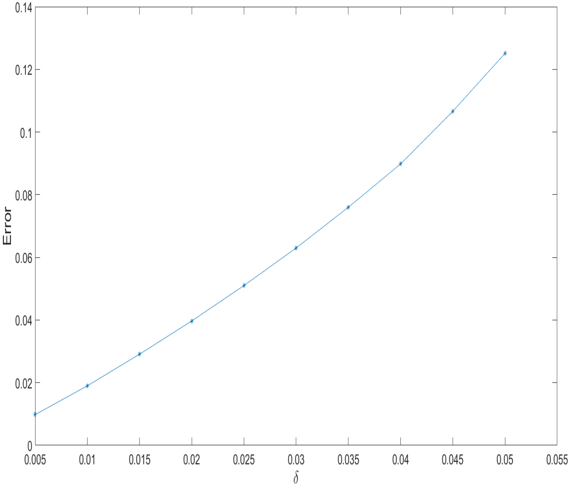


Figure: *Error for different values of δ*

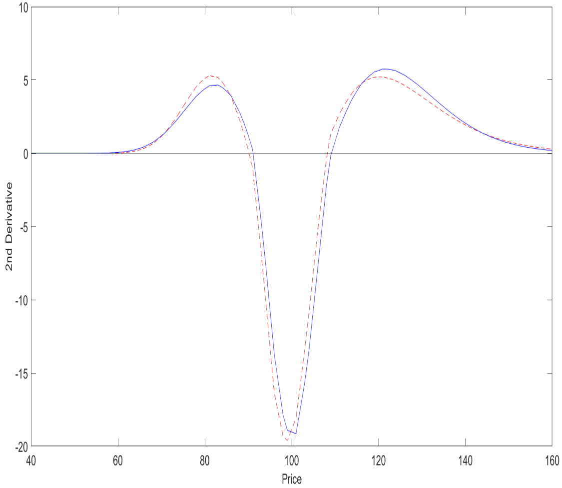


Figure: *The red curve marked with "- -" represents $\partial^2_{xx} P^\delta$; the blue curve represents $\partial^2_{xx} P_0$.*

New Trend in Financial Mathematics

Researchers are explorers and exploring never ends.

- Fractional Brownian motion (fBM), B. Mandelbrot and J. Van Ness, 1968.
  Fractional Brownian motions and its applications.
- Fractional stochastic volatility (FSV) model, F. Comte and E. Renault, 1998.
  Model log-volatility using fBM with Hurst parameter $H > \frac{1}{2}$.
- Rough fractional stochastic volatility (RFSV) model, J. Gatheral, T. Jaisson and M. Rosenbaum, 2014.
  - $H < \frac{1}{2}$
  - Remarkably consistent with financial time series data
  - Improved forecasts of realized volatility
- Rough Uncertain Volatility model? 2017?

Thank you!