

Portfolio optimization near horizon

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Problem Formulation

- **Problem:** Determine *optimal* investment strategy when trading between one risky and one riskless asset in trading horizon $[t, T]$ (for $0 \leq t < T$, T fixed).
 - ▶ *Optimal trading* maximizes expected utility of terminal wealth.
 - ▶ Under appropriate assumptions, study via associated Hamilton Jacobi Bellman PDE.
- **Market:** Stochastic volatility model.
 - ▶ One riskless asset B_t .
 - ▶ One risky asset S_t ; drift and diffusion depend on stochastic factor driven by correlated Brownian motion.

Overview

- Only assumptions on utility function will be on asymptotic behavior as wealth approaches 0 and ∞ .
- Well-posedness of associated HJB equation not established.
 - ▶ Nadtochiy & Zariphopoulou (2013) proved marginal HJB has a unique viscosity solution.
- **Goal:** Approximate optimal portfolio by building sub- and super-solutions to marginal HJB equation.
 - 1 Approximate **viscosity solution** of “marginal HJB equation”.
 - 2 Use above to approximate **value function**.
 - 3 Construct approximating **portfolio**.
 - 4 Results valid when $T - t$ small.

Model Assumptions

- $W. := (W^1, W^2) = 2$ -dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$; take natural filtration generated by $W.$, i.e.,

$$\mathcal{F}_t := \sigma(W_s : 0 \leq s \leq t), \quad 0 \leq t \leq T$$

- *Asset dynamics:*

$$\begin{cases} dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^1 \text{ (risky)} \\ dB_t = rB_t dt \text{ (riskless)} \end{cases}.$$

- *Stochastic factor dynamics:*

$$dY_t = b(Y_t) dt + a(Y_t) (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2).$$

- Risky asset return has stochastic volatility dependent on factor driven by an imperfectly correlated Brownian motion ($\rho \in (-1, 1)$).

Model Assumptions

- $\mu, \sigma \in C(\mathbb{R})$, with $\sigma > 0$.
- $b \in C^1(\mathbb{R})$, $\lambda, a \in C^2(\mathbb{R})$, where $\lambda(y) := \frac{\mu(y) - r}{\sigma(y)}$.
- For some constant C ,

$$|a| + \left| \frac{1}{a} \right| + |a'| + |a''| + |b| + |b'| + |\lambda| + |\lambda'| + |\lambda''| \leq C.$$

- π_t (π_t^0) denotes discounted amount of wealth invested in risky (riskless) asset
 - ▶ Only self-financing trading strategies considered.
 - ▶ Portfolios identified by π_t .
 - ▶ Wealth defined by $X_t^\pi := \pi_t + \pi_t^0$.

Model Assumptions

- *Investor's wealth dynamics:*

$$\begin{cases} dX_s^{t,x,\pi} = \sigma(Y_s)\pi_s(\lambda(Y_s) ds + dW_s^1), & t \leq s \leq T \\ X_t^{t,x,\pi} = x, & 0 < x < \infty \end{cases}.$$

- $U_T : (0, \infty) \rightarrow \mathbb{R}$ is the utility function that indicates the investor's risk preferences at time T .
 - ▶ Assume U_T strictly increasing, concave, in $C^5(\mathbb{R})$.
- Set $u(x) := U'_T(x)$, $R(x) := -\frac{d}{dx} \left(\frac{1}{2} \frac{u^2(x)}{u'(x)} \right)$.
 - ▶ Following assumptions on asymptotic behavior of $u(x)$, $R(x)$ ensure, for example, that $U_T(x) \sim \frac{x^{1-\gamma}}{1-\gamma}$ near 0 and ∞ for some positive $\gamma \neq 1$.

Assumptions on Asymptotic Behavior

- Assume the following about the asymptotic behavior of u, R for some fixed positive $\gamma \neq 1$:

Assumption

For some positive $\gamma \neq 1$,

- $0 < \inf_{x>0} (x^\gamma u(x)) \leq \sup_{x>0} (x^\gamma u(x)) < \infty$
- $0 < \inf_{x>0} (x^\gamma |R(x)|) \leq \sup_{x>0} (x^\gamma |R(x)|) < \infty$
- $0 < \inf_{x>0} \left(-x \frac{u'(x)}{u(x)} \right) \leq \sup_{x>0} \left(-x \frac{u'(x)}{u(x)} \right) < \infty$
- $0 < \inf_{x>0} (x^{1+\gamma} |R'(x)|) \leq \sup_{x>0} (x^{1+\gamma} |R'(x)|) < \infty$
- $0 < \inf_{x>0} (x^{2+\gamma} |u''(x)|) \leq \sup_{x>0} (x^{2+\gamma} |u''(x)|) < \infty$
- $0 < \inf_{x>0} (x^{2+\gamma} |R''(x)|) \leq \sup_{x>0} (x^{2+\gamma} |R''(x)|) < \infty$

Admissible Strategies

- Only interested in optimizing over a subset of all possible trading strategies.
- Denote by \mathcal{A} the set of admissible strategies which satisfy 1 – 4 in the below definition.

Definition

A self-financing strategy $\{\pi_s\}_{s \in [t, T]}$ is *admissible* if

- 1 π . is progressively measurable wrt $\{\mathcal{F}_t\}_{t \in [0, T]}$.
- 2 π . is locally square integrable.
- 3 $X_s^{t, x, \pi} > 0$ for all $s \in [t, T]$.
- 4 If $\gamma > 1$, $E \int_t^T (X_s^{t, x, \pi})^{-\gamma} (1 + \pi_s^2) ds < \infty$ for every $\gamma \geq 0$.

HJB Equation

- **Goal:** Approximate $\pi^* \in \mathcal{A}$ which satisfies

$$\begin{aligned} & E[U_T(X_T^{t,x,\pi^*}) | X_t^{t,x,\pi^*} = x, Y_t = y] \\ &= \operatorname{ess\,sup}_{\pi \in \mathcal{A}} E[U_T(X_T^{t,x,\pi}) | X_t^{t,x,\pi} = x, Y_t = y]. \end{aligned}$$

- **Value function:**

$$J(t, x, y) := \operatorname{ess\,sup}_{\pi \in \mathcal{A}} E[U_T(X_T^{t,x,\pi}) | X_t^{t,x,\pi} = x, Y_t = y].$$

- Dynamic programming gives the associated HJB PDE:

$$\begin{aligned} U_t + \max_{\pi \in \mathcal{A}} \left\{ \frac{1}{2} \sigma^2(y) \pi^2 U_{xx} + \pi (\sigma(y) \lambda(y) U_x + \rho \sigma(y) a(y) U_{xy}) \right\} \\ + \frac{1}{2} a^2(y) U_{yy} + b(y) U_y = 0 \end{aligned}$$

Marginal HJB Equation

- The maximizer is clearly given by

$$\pi^* = \frac{-\lambda(y)U_x - \rho a(y)U_{xy}}{\sigma(y)U_{xx}}.$$

- Substituting π^* into the HJB equation and differentiating wrt x gives the **marginal HJB equation**, which $V(t, x, y) := U_x(t, x, y)$ formally solves:

$$\begin{aligned} V_t + \frac{1}{2} \left(\frac{\lambda(y)V + \rho a(y)V_y}{V_x} \right)^2 V_{xx} - \frac{\lambda(y)V + \rho a(y)V_y}{V_x} \rho a(y)V_{xy} \\ + \frac{1}{2} a^2(y)V_{yy} - \lambda^2(y)V + (b(y) - \lambda(y)\rho a(y))V_y = 0. \end{aligned}$$

It has been established in Nadtochiy and Zariphopoulou (2013) that the marginal HJB equation has a unique viscosity solution, henceforth denoted $V(t, x, y)$.

Main Result (Approximating Viscosity Solution)

Theorem (Kumar-N.)

For some constant $C > 0$,

$$|V(t, x, y) - (u(x) + (T - t)\lambda^2(y)R(x))| \leq C(T - t)^2 x^{-\gamma}$$

as $(T - t) \searrow 0$.

Sketch of proof

Goal: Construct sub- and super-solutions \underline{V} and \bar{V} , respectively, to the marginal HJB equation so that

$$\underline{V}(t, x, y) \leq V(t, x, y) \leq \bar{V}(t, x, y).$$

Constructing Sub- and Super-Solutions

Proof (cont.)

Assume

$V(t, x, y) = V^{(0)}(x, y) + (T - t)V^{(1)}(x, y) + (T - t)^2 V^{(2)}(x, y) + \dots$
and substitute this into the marginal HJB equation. Grouping powers of $T - t$, we deduce from the $O(1)$ terms that

$$V^{(0)}(x, y) = u(x)$$

for all $(x, y) \in (0, \infty) \times \mathbb{R}$. Similarly, from the $O(T - t)$ terms, we find

$$V^{(1)}(x, y) = \lambda^2(y)R(x)$$

for all $(x, y) \in (0, \infty) \times \mathbb{R}$.

Constructing Sub- and Super-Solutions

Proof (cont.)

We now choose $\underline{v}^{(2)}(x, y)$ and $\bar{v}^{(2)}(x, y)$ such that

$$\underline{V}(t, x, y) = u(x) + (T - t)\lambda^2(y)R(x) + (T - t)^2 \underline{v}^{(2)}(x, y)$$

and

$$\bar{V}(t, x, y) = u(x) + (T - t)\lambda^2(y)R(x) + (T - t)^2 \bar{v}^{(2)}(x, y)$$

will be a sub-solution and super-solution, respectively. The $O((T - t)^2)$ terms suggest the formulas $\bar{v}^{(2)}(x, y) = Cx^{-\gamma}$ and $\underline{v}^{(2)}(x, y) := -\bar{v}^{(2)}(x, y)$ for some constant $C > 0$, yielding, for small enough $(T - t)$, that \underline{V} and \bar{V} sub- and super-solutions, respectively, of the marginal HJB equation.

Applying Comparison Principle

Proof (cont.)

Now we establish $\underline{V}(t, x, y) \leq V(t, x, y) \leq \bar{V}(t, x, y)$, which requires a previously established comparison principle for a related PDE, established in Nadtochiy and Zariphopoulou (2013).

Consider $W(t, z, y) := \log(V(t, e^z, y)) + \gamma z$, where $z = \log(x)$. W formally solves:

$$W_t + \epsilon \left[-\frac{1}{2} \frac{(\lambda + a\rho W_y^\epsilon)^2}{W_z - \gamma} \left(\frac{W_{zz}}{W_z - \gamma} - 1 \right) - \frac{1}{2} a^2 W_{yy} \right. \\ \left. + a\rho \frac{\lambda + a\rho W_y}{W_z - \gamma} W_{zy} + \frac{1}{2} \lambda^2 + (\lambda a\rho - b) W_y + \frac{1}{2} a^2 (\rho^2 - 1) (W_y)^2 \right] = 0$$

In fact, Nadtochiy and Zariphopoulou (2013) showed that W is a viscosity solution of this PDE.

Applying Comparison Principle

Proof (cont.)

Choose $(T - t)$ small enough (s.t. $\underline{V}, \bar{V} > 0$) and define

$$\underline{W}(t, z, y) = \log(\underline{V}(t, e^z, y)) + \gamma z$$

and

$$\bar{W}(t, z, y) = \log(\bar{V}(t, e^z, y)) + \gamma z.$$

Then \underline{W} and \bar{W} are bounded classical sub- and super-solutions of the above equation.

The comparison principle then implies $\underline{W} \leq W \leq \bar{W}$, giving $\underline{V} \leq V \leq \bar{V}$, i.e.,

$$\begin{aligned} u(x) + (T - t)\lambda^2(y)R(x) - c_2x^{-\gamma}(T - t)^2 &\leq V(t, x, y) \\ &\leq u(x) + (T - t)\lambda^2(y)R(x) + c_2x^{-\gamma}(T - t)^2, \end{aligned}$$

giving $|V(t, x, y) - (u(x) + (T - t)\lambda^2(y)R(x))| \leq c_2(T - t)^2x^{-\gamma}$, as desired.

Corollary (Approximating Value Function)

Corollary (Kumar-N.)

Let $J(t, x, y) = \operatorname{ess\,sup}_{\pi \in \mathcal{A}} E[U_T(X_T^{t,x,\pi}) | X_t^{t,x,\pi} = x, Y_t = y]$. For some constant $C > 0$,

$$\left| J(t, x, y) - \left(U_T(x) - (T - t) \frac{\lambda^2(y)}{2} \frac{U_T'(x)^2}{U_T''(x)} \right) \right| \leq C \frac{(T - t)^2}{1 - \gamma} x^{1-\gamma}$$

as $(T - t) \searrow 0$.

- **Remark.** Key ingredient in proof of corollary: $J(t, x, y)$ can be represented as the integral of the viscosity solution of the marginal HJB equation, a result due to Nadtochiy and Zariphopoulou (2013).

Approximating Portfolio

- The next lemma asserts the sub-solution of the marginal HJB equation generates a portfolio which is close to optimal; the case $\gamma \in (0, 1)$ is dealt with, while the case $\gamma \in (1, \infty)$ is analogous.

Lemma (Kumar-N.)

Suppose $\underline{U}(t, x, y) := U_T(0+) + \int_0^x \underline{V}(t, r, y) dr$ and

$$\underline{\pi}(t, x, y) := -\frac{\lambda(y)}{\sigma(y)} \frac{\underline{U}_x}{\underline{U}_{xx}} - \frac{\rho a(y)}{\sigma(y)} \frac{\underline{U}_{xy}}{\underline{U}_{xx}}.$$

Then

$$|U(t, x, y) - E[U_T(X_T^{t,x,\pi}) | \mathcal{F}_t]| \leq C \frac{(T-t)^2}{1-\gamma} x^{1-\gamma}$$

as $(T-t) \searrow 0$.

Proof of Lemma

Proof

Applying Ito's formula to $\underline{U}(s, X_s^{t,x,\pi}, Y_s)$ gives

$$\begin{aligned} & \underbrace{\underline{U}(T, X_T^{t,x,\pi}, Y_T) - \underline{U}(t, X_t^{t,x,\pi}, Y_t)}_{U_T(X_T^{t,x,\pi})} \\ &= \int_t^T \left[\underline{U}_t + \sigma \pi \lambda \underline{U}_x + b \underline{U}_y + \frac{1}{2} \sigma^2 \pi^2 \underline{U}_{xx} + \sigma \pi a \rho \underline{U}_{xy} + \frac{1}{2} a^2 \underline{U}_{yy} \right] ds \\ & \quad + \int_t^T (\sigma \pi \underline{U}_x + a \rho \underline{U}_y) dW_s^1 + \int_t^T a \sqrt{1 - \rho^2} \underline{U}_y dW_s^2. \end{aligned}$$

Taking conditional expectation causes the martingale terms to drop out, giving:

Proof of Lemma

Proof (cont.)

$$\begin{aligned} & E[\underline{U}(T, X_T^{t,x,\pi}, Y_T) | \mathcal{F}_t] - \underline{U}(t, X_t^{t,x,\pi}, Y_t) \\ &= \int_t^T E \left[\underline{U}_s + \sigma \pi \lambda \underline{U}_x + b \underline{U}_y + \frac{1}{2} \sigma^2 \pi^2 \underline{U}_{xx} + \sigma \pi a \rho \underline{U}_{xy} + \frac{1}{2} a^2 \underline{U}_{yy} | \mathcal{F}_s \right] ds \\ &\geq 0 \end{aligned}$$

where the last inequality follows from \underline{U} being a sub-solution of the HJB equation. Thus,

$$E[U_T(X_T^{t,x,\pi}) | \mathcal{F}_t] \geq \underline{U}(t, X_t^{t,x,\pi}, Y_t).$$

Thus, we have

$$\underline{U}(t, X_t^{t,x,\pi}, Y_t) \leq E[U_T(X_T^{t,x,\pi}) | \mathcal{F}_t] \leq \operatorname{ess\,sup}_{\pi \in \mathcal{A}} E[U_T(X_T^{t,x,\pi}) | \mathcal{F}_t] = J(t, x, y).$$

Proof of Lemma

Proof (cont.)

By the previous theorem, $0 \leq J(t, x, y) - \underline{U}(t, x, y) \leq C \frac{(T-t)^2}{1-\gamma} x^{1-\gamma}$, and this together with the previous inequality give

$$|J(t, x, y) - E[U_T(X_T^{t,x,\pi})|\mathcal{F}_t]| \leq C \frac{(T-t)^2}{1-\gamma} x^{1-\gamma},$$

as desired.

Closed Form Formula

- **Remark.** We have found a formula for a portfolio $\underline{\pi}$ under which the expected utility of terminal wealth is close to the maximal expected utility under the optimal portfolio.
 - ▶ In practice, $\underline{\pi}$ cannot be computed since it is generated by the sub-solution

$$\underline{U}(t, x, y) = U_T(0+) + \int_0^x u(w) + (T - t)\lambda^2 R(w) - Cw^{-\gamma}(T - t)^2 dw$$

the formula of which has an unspecified constant C .

- **Idea:** Consider the portfolio $\hat{\pi}$ generated by

$$\hat{U}(t, x, y) = U_T(0+) + \int_0^x u(w) + (T - t)\lambda^2(y)R(w) dw.$$

Closed Form Formula

Lemma (Kumar-N.)

If $\underline{\pi}$ is generated by the sub-solution \underline{U} and $\hat{\pi}$ is generated by \hat{U} , then

$$|\underline{\pi} - \hat{\pi}| = O((T - t)^2)O(1 + x)$$

as $(T - t) \searrow 0$.

- Thus, for each fixed x , $\hat{\pi}$ approximates $\underline{\pi}$ when the time to horizon is small.
- We next illustrate graphically the degree of approximation through an example from Chacko and Viceira (2005).

Example

- We consider the model found in Chacko and Viceira (2005), in which they estimated the parameters from real market data.
- **Recall:** The risky asset's price evolves as

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^1,$$

and the level of the stochastic factor evolves as

$$dY_t = b(Y_t) dt + a(Y_t)(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2).$$

- In this example, take $\mu(y) = \mu$, $\sigma(y) = \frac{1}{\sqrt{y}}$, $b(y) = m - y$, and $a(y) = \beta\sqrt{y}$, where μ, m, β are constants.
- Set $\mu = 0.0811$, $m = 27.9345$, $\beta = 1.12$, $y = 27.9345$, $T = 2$, $\gamma = 3$, and $\rho = 0.5241$.
- Consider the utility function $U_T(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Example

- The value function is given by

$$U(t, x, y) = -\frac{1}{2x^2} e^{\left(\frac{\gamma}{\gamma+(1-\gamma)\rho^2}\right)(yA(t,T)+B(t,T))} \approx -\frac{0.485022}{x^2}$$

where $t = 1.5$ in the approximation.

- Similarly,

$$\hat{U}(t, x, y) = U_T(x) - (T - t) \frac{\lambda^2(27.9345)}{2} \frac{U'_T(x)^2}{U''_T(x)} \approx -\frac{0.484689}{x^2}.$$

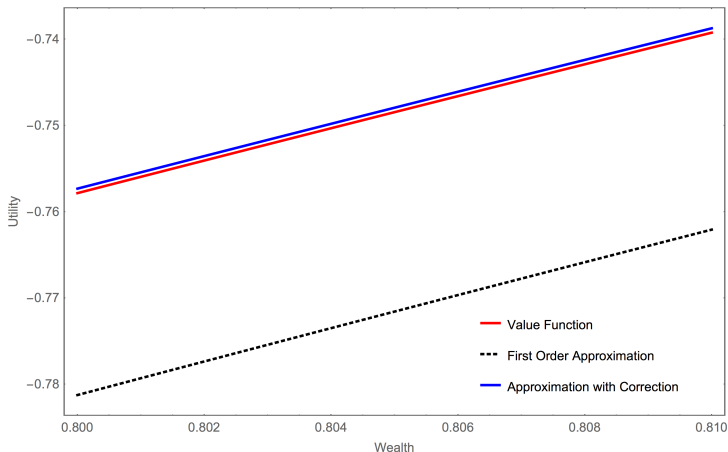
- The respective portfolios generated by the above functions are given by

$$\pi_U(x) \approx 0.750482x$$

and

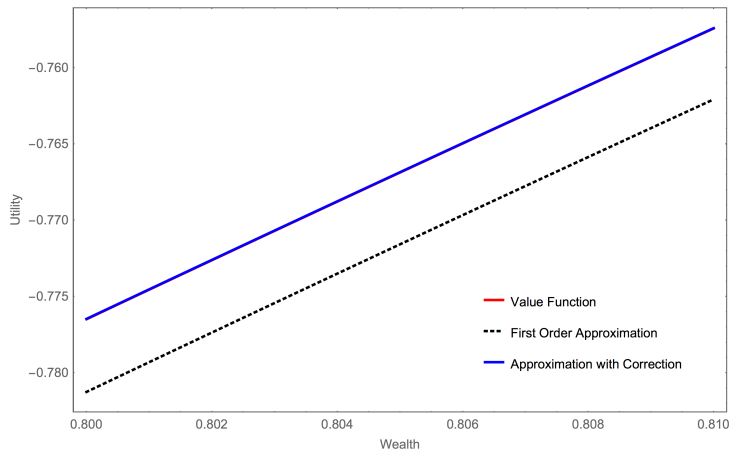
$$\hat{\pi}(x) \approx 0.748982x.$$

Example



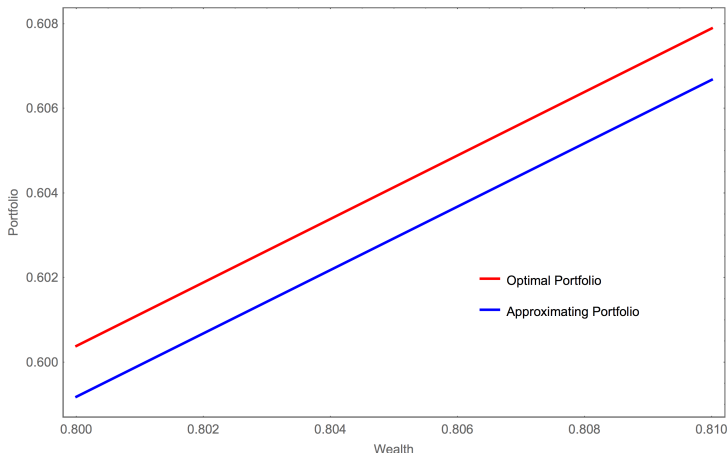
The value function plotted against the first order approximation and the approximation with correction. $t = 1.5$, $T = 2$.

Example



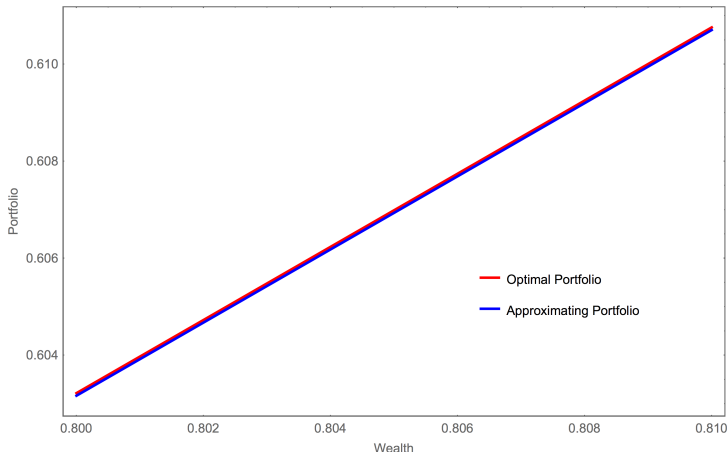
The value function plotted against the first order approximation and the approximation with correction. $t = 1.9$, $T = 2$.

Example



The optimal portfolio plotted against the approximating portfolio.
 $t = 1.5, T = 2$.

Example



The optimal portfolio plotted against the approximating portfolio.
 $t = 1.9, T = 2$.

Example

- **Remark.** In the previous example, $\lambda(y) = \frac{\mu-r}{\sigma(y)}$ is unbounded, against the initial assumptions of our work. That our results can still be applied to this example to illustrate the approximation suggests that we may be able to weaken some assumptions.

References



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Thank you!