Heterogeneous Preferences and General Equilibrium in Financial Markets

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- Characterize equilibrium as system of uncoupled ode’s ⇒ Dimension of state space does not depend on number of types.
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- In this paper: study the effects of these differences on financial markets. Does the number of types affect outcomes?
- Characterize equilibrium as system of uncoupled ode’s ⇒ Dimension of state space does not depend on number of types.
- Results:
  - Relationship between variance in EIS and equity risk premium/risk-free rate puzzles.
  - Number of types is important.
  - Returns are predictable as function of dividends.
  - Excess volatility and volatility smile as a low dividend corresponds to high volatility and vice-versa.
How many preference types? Dumas [1989], Bhamra and Uppal [2014], Chabakauri [2015], and Gârleanu and Panageas [2015] focus on two agents. Question remains: Do results generalize quantitatively to many agents?

Dynamics vs. asymptotics? This paper most closely resembles Cvitanić, Jouini, Malamud, and Napp [2011] and Chabakauri [2015], but I add the explicit study of the effects of changes in the distribution of preferences on short-run dynamics for an arbitrary number of types.

Asset pricing "puzzles" exist in this model, namely the risk-free rate puzzle (Weil [1989]), the equity risk premium puzzle (Mehra and Prescott [1985]), and the volatility smile (Fouque, Papanicolaou, Sircar, and Sølna [2011]), as well as leverage cycles (Geanakoplos [2010]) and returns predictability (Campbell and Shiller [1988a,b], Mankiw [1981]).
Set Up

- $i \in \{1, 2, ..., N\}$ agents who are price takers
- CRRA instantaneous utility functions, heterogeneous in RRA parameter $\gamma_i$

$$U_i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1 - \gamma_i} \quad \forall i \in \{1, 2, ..., N\}$$

- Assume $\gamma \in [1, \bar{\gamma})$ for exposition.
- Agents preference parameter, $\gamma_i$, and their initial wealth, $x_i$, are drawn from a joint distribution

$$(\gamma_i, x_i) \sim f(\gamma, x)$$
Risk is driven by a single Brownian motion $W_t$.

Trade risky shares, in $N$ Lucas trees paying dividends, $D(t)$, which follows a geometric Brownian motion:

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t$$

Price of risky and risk free shares, $S_t$ and $S_t^0$, determined in equilibrium:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

$$\frac{dS_t^0}{S_t^0} = r_t dt$$
Budget Constraints

By standard derivations (Merton [1969] or Karatzas and Shreve [1998]) we can arrive at an SDE describing the evolution of agents' wealth $X_{it}$:

$$dX_{it} = \left[ X_{it} \left( r_t + \pi_{it} \left( \mu_t + \frac{D_t}{S_t} - r_t \right) \right) - c_{it} \right] dt + \pi_{it} X_{it} \sigma_t dW_t$$

Assume non-negative wealth:

$$X_{it} \geq 0, \quad \forall \ t \in [0, \infty) \quad \text{a.s.}$$

Markets clear:

$$\frac{1}{N} \sum_i c_{it} = D_t, \quad \frac{1}{N} \sum_i (1 - \pi_{it}) X_{it} = 0, \quad \frac{1}{N} \sum_i \pi_{it} X_{it} = S_t$$
Following the martingale method of Harrison and Pliska [1981], Karatzas, Lehoczky, and Shreve [1987], define the stochastic discount factor (SDF) $H_t$ as

$$\frac{dH_t}{H_t} = -r_t dt - \theta_t dW_t$$

where

$$\theta_t = \frac{\mu_t + \frac{D_t}{S_t} - r_t}{\sigma_t}$$
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FOC implies:

$$c_{it} = \left( \Lambda_i e^{\rho t} H_t \right)^{-\frac{1}{\gamma_i}}$$
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Using market clearing gives consumption weights:

$$c_{it} = \omega_{it} D_t$$

where

$$\omega_{it} = \frac{N \left( \Lambda_i e^{\rho t} H_t \right)^{-\frac{1}{\gamma_i}}}{\sum_j \left( \Lambda_j e^{\rho t} H_t \right)^{-\frac{1}{\gamma_j}}}$$
Proposition

The interest rate and market price of risk are fully determined by the sufficient statistics $\xi_t = \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_{it}}{\gamma_i}$ and $\phi_t = \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_{it}}{\gamma_i^2}$ such that

$$r_t = \rho + \frac{\mu D}{\xi_t} - \frac{1}{2} \frac{\xi_t + \phi_t}{\xi_t^3} \sigma_D^2$$

$$\theta_t = \frac{\sigma_D}{\xi_t}$$
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\[
\begin{align*}
    r_t &= \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2} \frac{\xi_t + \phi_t}{\xi_t^3} \sigma_D^2 \\
    \theta_t &= \frac{\sigma_D}{\xi_t}
\end{align*}
\]

\( r_t \) can be rewritten as the rate that would prevail under a single agent with time varying preferences and an extra term

\[
\begin{align*}
    r_t &= \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2} \frac{\xi_t + 1}{\xi_t^2} \sigma_D^2 - \frac{1}{2} \frac{1}{\xi_t} \left( \frac{\phi_t}{\xi_t^2} - 1 \right) \sigma_D^2
\end{align*}
\]
Solution: CRRA Representative

- Can we match the instantaneous RFR and MPoR with a representative CRRA agent?
- Define $\gamma_t$ and $\theta_t$ such that

$$r_t = \rho + \gamma_t \mu_D - \gamma_t (1 + \gamma_t) \frac{\sigma_D^2}{2}$$

$$\theta_t = \sigma_D \theta_t$$
Can we match the instantaneous RFR and MPoR with a representative CRRA agent?

Define \( \gamma_{rt} \) and \( \gamma_{\theta t} \) such that

\[
rt = \rho + \gamma_{rt} \mu_D - \gamma_{rt} (1 + \gamma_{rt}) \frac{\sigma^2_D}{2}
\]

\[
\theta_t = \sigma_D \gamma_{\theta t}
\]

It can be shown that \( \gamma_{rt} < \gamma_{\theta t} \).

\[
\begin{align*}
\gamma & \quad \gamma_{rt} & \quad \gamma_{\theta t} & \quad \bar{\gamma} \\
\text{Investor} & \quad \text{Divestor} & \quad \text{Borrower} & \quad \text{Lender} \\
\text{Leveraged Investor} & \quad \text{Diversifying Investor} & \quad \text{Saving Divestor}
\end{align*}
\]
Solution: Consumption Weight Dynamics

- Consumption weights follow Ito processes:

\[
\frac{d\omega_{it}}{\omega_{it}} = \mu_{it} dt + \sigma_{it} dW_t
\]

Volatility in consumption weights depends on relative position to the marginal agent:

\[
\sigma_{it} = \sqrt{t} \Rightarrow \omega_{it} \sim t
\]

Some agents are buying low and selling high, while others are not. Can find consumption weights as functions of the dividend:

\[
\omega_{it} = f_i(D_t), \text{ such that } f_i(\cdot) \text{ is an implicit function}
\]
Consumption weights follow Ito processes:

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Volatility in consumption weights depends on relative position to the marginal agent:

\[ \sigma_{it} = \theta_t \left( \frac{1}{\gamma_i} - \xi_t \right) \]
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- Some agents are buying low and selling high, while others are not.
- Can find consumption weights as functions of the dividend. \( \omega_{it} = f_i(D_t) \), such that \( f_i(\cdot) \) is an implicit function

\[
\frac{1}{N} \sum_j \lambda_{ji}^{-1} f_i(z) \frac{\gamma_i}{\gamma_j} z \frac{\gamma_j - \gamma_i}{\gamma_j} = 1 \text{ where } \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i}
\]
Solution: Wealth/Consumption Ratios

Wealth/consumption ratios $V_i(D) = X_{it}/c_{it}$ satisfy a system of uncoupled ODE’s:

$$0 = 1 + \frac{\sigma_D^2 D_t^2}{2} V''_i(D_t) + \left[ \frac{1 - \gamma_i}{\gamma_i} \theta_t \sigma_D + \mu_D \right] D_t V'_i(D_t)$$

$$+ \left[ (1 - \gamma_i) r_t - \rho + \frac{1 - \gamma_i}{2 \gamma_i} \theta_t^2 \right] \frac{V_i(D_t)}{\gamma_i}$$

with appropriate boundary conditions. Portfolios are given by:

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t} \left( \gamma_i \sigma_D D_t \frac{V'_i(D_t)}{V_i(D_t)} + \theta_t \right)$$
Solution: Wealth/Consumption Ratios

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$$+ \left[ (1 - \gamma_i) r_t - \rho + \frac{1-\gamma_i}{2 \gamma_i} \theta_t^2 \right] \frac{V_i(D_t)}{\gamma_i}$$

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$$\pi_{it} = \frac{1}{\gamma_i \sigma_t} \left( \gamma_i \sigma_D D_t \frac{V'_i(D_t)}{V_i(D_t)} + \theta_t \right)$$

- Portfolios exhibit myopic and hedging demand, where hedging demand is always positive.
Volatility is given by:

$$\sigma_t = \sigma_D \left( 1 + D_t \frac{S'_t(D_t)}{S_t(D_t)} \right)$$

where the price/dividend ratio $S_t(D_t)$ satisfies

$$S_t(D_t) = \frac{1}{N} \sum_i V_i(D_t) \omega_{it}$$

- Returns determined by $1/S_t(D_t)$, which is negatively correlated with $W_t \Rightarrow$ Campbell and Shiller [1988a,b], Mankiw [1981].
Solution: Volatility and P/D Ratio

Volatility is given by:

$$\sigma_t = \sigma_D \left( 1 + D_t \frac{S_t'(D_t)}{S_t(D_t)} \right)$$

where the price/dividend ratio $S_t(D_t)$ satisfies

$$S_t(D_t) = \frac{1}{N} \sum_i V_i(D_t) \omega_{it}$$

- Returns determined by $1/S_t(D_t)$, which is negatively correlated with $W_t \Rightarrow$ Campbell and Shiller [1988a,b], Mankiw [1981].
- Model exhibits excess volatility.
Numerical Solution

- Many possibilities, here focus on the effect of number of types.
  - Underlying research question: Is two types sufficient?
  - If so, this additional machinery unnecessary!
- Fix $\gamma_i \sim Uni(1.5, 10.0)$ and change the number of evenly spaced types.
- Fix $\mu_D = 0.01, \sigma_D = 0.032, \rho = 0.01$. (Chosen to match Chabakauri [2015].)
- Results: Number of types affects level and slope, but not direction of effects.
Changing the number of agents changes financial variables.

More types generates higher interest rate and MPoR.
Increased volatility and negative correlation $\Rightarrow$ volatility smile.

Negative correlation $D/S$ and $D$ as in Campbell and Shiller [1988a,b] $\Rightarrow$ predictable stock prices.

Predictability generated by comovement between SDF and consumption as in Mankiw [1981].
Fall in $D$ implies a rise in leverage $\Rightarrow$ counter-cyclical leverage cycles.

Opposite of that assumed/produced in the literature on beliefs generated cycles (Geanakoplos [2010]).

Complete market allows agents to leverage up in order to smooth consumption.
Given we are interested in heterogeneous preferences, we should consider the modeling choice of how many types.

Heterogeneous preferences generate dynamics that match real world data: volatility smile, falling interest rates, predictability of returns, leverage cycles.

Can partially explain several asset pricing puzzles (risk-free rate puzzle, equity risk premium puzzle, predictability of stock returns).

Second moment of distribution of preferences matter for RFR and ERP puzzles!

Looking forward, the introduction of portfolio constraints may provide even better results, in particular for term structure and direction of leverage cycles.
Thanks!


Additional Materials
The control problem is time inconsistent and non-markovian! Can apply the martingale method (or Girsonov theory) to transform dynamic to static. Define the stochastic discount factor as

\[ H_0(t) = \exp \left( -\int_0^t r(u)du - \int_0^t \theta(u)dW(u) - \frac{1}{2} \int_0^t \theta(u)^2 du \right) \]

where

\[ \theta(t) = \frac{\mu_s(t) + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)} \]

represents the market price of risk. This implies that the stochastic discount factor also follows a diffusion of the form

\[ \frac{dH_0(t)}{H_0(t)} = -r(t)dt - \theta(t)dW(t) \]
Using the stochastic discount factor, we can rewrite each agent’s dynamic problem as a static one beginning at time $t = 0$

$$\max \{c^i(u)\}_{u=0}^{\infty} \quad \mathbb{E} \int_0^\infty e^{-\rho u} \frac{c^i(u)^{1-\gamma_i} - 1}{1 - \gamma_i} \, du$$

s.t. \quad \mathbb{E} \int_0^\infty H_0(u)c^i(u) \, du \leq x_i$$

First order condition by calculus of variations:

$$c^i(t) = (\Lambda_i e^{\rho t} H_0(t))^{-\frac{1}{\gamma_i}}$$
Solution: Matching Equity Risk Premium

- Fix $r_0 = 0.03, \sigma_D = 0.032, \rho = 0.02$, and plot the values of $\xi_t$ and $\phi_t$ which give $\theta$ for different values of $\mu_D$. (plot a)

![Diagram a](image1.png)

(a) Fix $r_0 = 0.03, \sigma_D = 0.032, \rho = 0.02$, and plot the values of $\xi_t$ and $\phi_t$ which give $\theta$ for different values of $\mu_D$. (plot a)

(b) Consider three agents with preference parameters fixed to $(1, 2, 3) = (1.1, 1.0, 1.8)$. Could these agents produce observed ERP? (plot b)
Solution: Matching Equity Risk Premium

- Fix $r_0 = 0.03$, $\sigma_D = 0.032$, $\rho = 0.02$, and plot the values of $\xi_t$ and $\phi_t$ which give $\theta$ for different values of $\mu_D$. \textbf{(plot a)}
- Consider three agents with preference parameters fixed to $(\gamma_1, \gamma_2, \gamma_3) = (1.1, 10, 18)$. Could these agents produce observed ERP? \textbf{(plot b)}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plot.png}
\caption{(c) and (d) show the graphs for $\xi_*$ and $\phi_*$, respectively.}
\end{figure}
**Proposition**

*Assuming consumption weights also follow a geometric Brownian motion such that*

\[
\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t)
\]

*then* \(\mu_{\omega^i}(t)\) *and* \(\sigma_{\omega^i}(t)\) *are given by:*

\[
\mu_{\omega^i}(s) = (r(t) - \rho) \left( \frac{1}{\gamma_i} - \xi(t) \right) + \theta(t)^2 \left[ \left( \frac{1}{\gamma_i^2} - \phi(t) \right) - 2\xi(t) \left( \frac{1}{\gamma_i} - \xi(t) \right) + \frac{1}{\gamma_i} - \xi(t) \right]
\]

\[
\sigma_{\omega^i}(t) = \theta(t) \left( \frac{1}{\gamma_i} - \xi(t) \right)
\]
Recall the definition of $\omega^i(t) = \omega(\gamma^i, x^i, t)$ and consider the limit in $N$:

$$
\frac{N (\Lambda_i e^{\rho t} H_0(t))^{-1}}{\sum_{j=1}^N (\Lambda_j e^{\rho t} H_0(t))^{-1}} \xrightarrow{N \to \infty} \frac{(\Lambda(\gamma, x) e^{\rho t} H_0(t))^{-1}}{\int (\Lambda(\gamma, x) e^{\rho t} H_0(t))^{-1} dF(\gamma, x)}
$$

by the law of large numbers, which implies $\omega(\gamma^i, x^i, t) \xrightarrow{N \to \infty} \omega(\gamma, x, t)$. A similar result holds for $\xi(t)$ and $\phi(t)$, as well as for financial variables (e.g. $r(t)$, $\theta(t)$, etc.).
In the continuous types case, \( \omega(\gamma, x, t) \) is the Radon-Nikodym derivative of the initial distribution \( F(\gamma, x) \) with respect to another, stochastic distribution:

\[
1 = \int \omega(\gamma, x, t) dF(\gamma, x)
= \int \frac{dG(\gamma, x, t)}{dF(\gamma, x)} dF(\gamma, x)
= \int dG(\gamma, x, t)
\]

Then \( \omega(\gamma, x, t) \) represents the dynamics of the infinite dimensional, Banach valued random process \( G(\gamma, x, t) \).

Is this the optimal transport? Can this be thought of as the solution to the Munge problem?
It can be shown that if one is attempting to match the continuous types approximation with a histogram (which is equivalent to discrete types), the best one can do is

\[ G(A, 0) = \int_A \omega(\gamma, x, 0) f(\gamma, x) d\gamma dx = \int_A \frac{1}{(\bar{\gamma} - \gamma)(\bar{x} - x)} d\gamma dx \]

That is, one could only match an initial condition where the product \( \omega(\gamma, x, 0)f(\gamma, x) \) is a uniform distribution.

The continuous types model allows for a greater amount of freedom with less computational cost.
The preference levels which clear the market are given by

\[ \gamma_r(t) = \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\left( \frac{\mu_D}{\sigma_D^2} \right)^2 - \frac{\mu_D}{\sigma_D^2} \left( 1 + \frac{2}{\xi(t)} \right) + \frac{\xi(t) + \phi(t)}{\xi(t)^3}} + \frac{1}{4} \]

\[ \gamma_\theta(t) = \frac{1}{\xi(t)} \]