Stochastic Control with nonlinear expectations

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Markovian stochastic control problems on a given horizon of time T can typically be written as

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}[\int_0^T g(\alpha_s, X_s^\alpha) ds + h(X_T^\alpha)],$$

where $\mathcal{A}$ is a set of admissible control processes α_s, and (X_s^α) is a controlled jump diffusion process. The random variable $h(X_T^\alpha)$ may represent a terminal reward and $g(\alpha_s, X_s^\alpha)$ an instantaneous reward process. Using dynamic programming principle and under appropriate assumptions, the associated value function of the stochastic control problem can then be characterized as the solution, in a certain sense, of a Hamilton-Jacobi-Bellman equation.

We are interested here in generalizing these results to the case when the linear expectation $\mathbb{E}$ is replaced by a nonlinear expectation induced by a Backward Stochastic Differential Equation (BSDE). Typically, such problems in the Markovian case can be formulated as

$$\sup_{\alpha \in \mathcal{A}} \mathcal{E}^\alpha_{0,T}[h(X_T^\alpha)],$$

where $\mathcal{E}^\alpha$ is the nonlinear conditional expectation associated with a BSDE with jumps with controlled driver $f(\alpha_t, X_t^\alpha, y, z, k)$. Note that when the driver f does not depend on the solution of the BSDE, that is when $f(\alpha_t, X_t^\alpha, y, z, k) \equiv g(\alpha_t, X_t^\alpha)$, then we are back to the classical linear expectation case.

In this talk, we shall consider the case when there is an additional control in the form of a stopping time. We shall thus study mixed generalized optimal control/stopping problems of the form

$$\sup_{\alpha \in \mathcal{A}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^\alpha_{0,\tau}[h(X_\tau^\alpha)],$$

where $\mathcal{T}$ denotes the set of stopping times.

We first establish a dynamic programming principle (DPP) for our mixed problem with f-expectation. This requires some specific techniques of stochastic analysis and BSDEs to handle measurability and other issues due to the nonlinearity of the expectation. The value function of the problem is not a solution of a reflected BSDE andhence the DPP dos not follow from the flow property forreflected BSDEs. The method we propose allows us to handle the case when the value function is not measurable, thus leading to a *weak* dynamic programming principle. Using this principle and properties of reflected BSDEs, we then prove that the value function of our mixed problem is a viscosity solution of a generalized Hamilton-Jacobi-Bellman

(HJB) variational inequality. Uniqueness of the viscosity solution is obtained under additional assumptions. Illustrating examples in mathematical finance are provided.

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