Some Results on Time-Inconsistent Optimal Control Problems

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Outline

1. Introduction: Time-Consistency
2. Time-Inconsistent Problems
3. Equilibrium Strategies
4. Open Problems

1. Introduction: Time-Consistency

Optimal Control Problem: Consider

$$\begin{cases} \dot{X}(s) = b(s, X(s), u(s)), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with (scalar) cost functional

$$J(t, x; u(\cdot)) = h(X(T)) + \int_t^T g(s, X(s), u(s))ds,$$

where

$$\mathcal{U}[t, T] = \{u : [t, T] \to U \mid u(\cdot) \text{ is measurable } \}.$$

Problem (C). For given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)) \equiv V(t, x).$$

Bellman Optimality Principle: For any $\tau \in [t, T]$,

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \Big[\int_t^\tau g(s, X(s), u(s))ds + V\big(\tau, X(\tau; t, x, u(\cdot))\big)\Big].$$

Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be optimal for $(t,x) \in [0, T) \times \mathbb{R}^n$.

$$\begin{aligned} V(t,x) &= J(t,x;\bar{u}(\cdot)) = \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds \\ &\qquad + J\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot)); \bar{u}(\cdot)\big|_{[\tau,T]}\big) \\ &\geq \int_t^\tau g(s, \bar{X}(s), \bar{u}(s))ds + V\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot))\big) \\ &\geq \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \int_t^\tau g(s, X(s), u(s))ds \\ &\qquad + V\big(\tau, X(\tau; t, x, u(\cdot))\big) = V(t,x). \end{aligned}$$

Thus, all the equalities hold.

Consequently,

$$\begin{aligned} J\big(\tau, \bar{X}(\tau); \bar{u}(\cdot)\big|_{[\tau,T]}\big) &= V(\tau, \bar{X}(\tau)) \\ &= \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J\big(\tau, \bar{X}(\tau); u(\cdot)\big), \quad \text{a.s.} \end{aligned}$$

Hence, $\bar{u}(\cdot)\big|_{[\tau,T]} \in \mathcal{U}[\tau, T]$ is **optimal** for $\big(\tau, \bar{X}(\tau; t, x, \bar{u}(\cdot))\big)$.

This is called the **time-consistency** of Problem (C).

Definition. A problem involving a decision-making is said to be **time-consistent** if

an **optimal** decision made at a given time t
will remain **optimal** at any time $s > t$.

If the above is not the case, the problem is said to be **time-inconsistent**.

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If the problem under consideration is time-consistent, then once an optimal decision is made, we will not regret afterwards!

If the whole world is **time-consistent**,

then the things are too **ideal**, the life will be much **easier**!

But, it might also be a little or too **boring**

(exciting to have some challenges)!

Fortunately (unfortunately?), the life is not that ideal!

(Challenges are around!)

Time-inconsistent problems exist almost everywhere!

2. Time-Inconsistent Problems

In reality, problems are **hardly time-consistent**:

An optimal decision/policy made at time t, more than often, will not stay optimal, thereafter.

Main reason: When building the model, describing the utility/cost, etc., the following are used:

subjective Time-Preferences and

subjective Risk-Preferences.

- **Time-Preferences:**

Most people do not discount exponentially! Instead, they over discount on the utility of immediate future outcomes.

* What if a car in front not moving **2 seconds** after the light turned green? (Give a horn!)
* Plan to finish a job within next week (Will you finish it **Monday**? or **Friday**?)

* Shopping using credit cards (meet **immediate** satisfaction)

* Unintentionally pollute the environment due to over-development

....

Immediate utility weighs heavier!

Annual rate is $r = 10\%$

Option (A): Get \$100 today (5/17/2017).

Option (B): Get \$105 ($> 100(1 + \frac{r}{12})$) on 6/17/2017.

Option (A′): Get \$110 ($= 100 \times 1.10$) on 5/17/2018.

Option (B′): Get \$115.50 ($> 110(1 + \frac{r}{12})$) on 6/17/2018.

For a **time-consistent** person,

$$\text{(A)} \asymp \text{(A}'\text{)}, \qquad \text{(B)} \asymp \text{(B}'\text{)},$$
$$\text{(B)} \succ \text{(A)}, \qquad \text{(B}'\text{)} \succ \text{(A}'\text{)}.$$

However, for an **uncertainty-averse** person,

$$\text{(A)} \succ \text{(B)}, \qquad \text{(B}'\text{)} \succ \text{(A}'\text{)}.$$

Magnifying the example:

Option (A): Get \$1M today (5/17/2017).

Option (B): Get \$1.05M ($> 1\text{M}(1 + \frac{r}{12})$) on 6/17/2017.

Option (A′): Get \$1.1M ($= 1\text{M} \times 1.10$) on 5/17/2017.

Option (B′): Get \$1.155M($> 1.1\text{M}(1 + \frac{r}{12})$) on 6/17/2017.

For an **uncertainty-averse** person,

$$(\text{A}) \succ (\text{B}), \qquad (\text{B}') \succ (\text{A}').$$

The feeling is stronger?

More rational in the farther future.

Exponential discounting: $\lambda_e(t) = e^{-rt}, \quad r > 0$ — discount rate

Hyperbolic discounting: $\lambda_h(t) = \frac{1}{1+kt}$ — a hyperbola

If let $k = e^r - 1$, i.e., $e^{-r} = \lambda_e(1) = \lambda_h(1) = \frac{1}{1+k}$, then

$$\lambda_e(t) = e^{-rt} = \frac{1}{(1+k)^t}, \qquad \lambda_h(t) = \frac{1}{1+kt}.$$

For $t \sim 0$, $t \mapsto \frac{1}{1+kt}$ decreases faster than $t \mapsto \frac{1}{(1+k)^t}$:

$$\lambda_h'(0) = -k < -\ln(1+k) = \lambda_e'(0),$$

Hyperbolic discounting actually appears in **people's behavior**.

* D. Hume (1739), "A Treatise of Human Nature"

"**Reason is, and ought only to be the slave of the passions.**"

People's actions/behaviors are due to their **passions**.

* A. Smith (1759), "The Theory of Moral Sentiments"

Utility is not intertemporally sparable but rather that past and future experiences, jointly with current ones, provide current utility.

Mathematically, one should have

$$U(t, X(t)) = f(U(t-r, X(t-r)), U(t+\tau, X(t+\tau)),$$

where $U(t, X)$ is the utility at (t, X).

Generalized Merton Problem

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s)dW(s), \\ X(t) = x. \end{cases}$$

$$J(t,x;u(\cdot),c(\cdot)) = \mathbb{E}_t\Big[\int_t^T \nu(t,s)c(s)^\beta(s)ds + \rho(t)X(T)^\beta\Big],$$

with $\beta \in (0,1)$. Classical case:

$$\nu(t,s) = e^{-\delta(s-t)}, \quad \rho(t) = e^{-\delta(T-t)}, \qquad 0 \le t \le s \le T.$$

Problem. Find $(\bar{u}(\cdot), \bar{c}(\cdot))$ to maximize $J(t,x;u(\cdot),c(\cdot))$.

For given $t \in [0, T)$, optimal solution:

$$\begin{cases} \bar{u}^t(s) = \dfrac{(\mu - r)\bar{X}^t(s)}{\sigma^2(1-\beta)}, \\ \bar{c}^t(s) = \dfrac{\nu(t, s)^{\frac{1}{1-\beta}} \bar{X}^t(s)}{e^{\frac{\lambda}{1-\beta}(T-s)} \rho(t)^{\frac{1}{1-\beta}} + \int_s^T e^{\frac{\lambda}{1-\beta}(\tau - s)} \nu(t, \tau)^{\frac{1}{\beta}} d\tau} \end{cases}$$

$$\lambda = \frac{[2r\sigma^2(1-\beta) + (\mu - r)^2]\beta}{2\sigma^2(1-\beta)}$$

It is **time-inconsistent**.

* Palacious–Huerta (2003), survey on history

* Strotz (1956), Pollak (1968), Laibson (1997), ...

* Finn E. Kydland and Edward C. Prescott, (1977)
  (2004 Nobel Prize winners)
  (classical **optimal control theory** not working)

* Ekeland–Lazrak (2008)

* Yong (2011, 2012) (Multi-person **differential games**)

* Wei–Yong–Yu (2017) (**recursive cost functional** case)

* Karnam–Ma–Zhang (2017)

- **Risk-Preferences:**

Consider two investments whose returns are: R_1 and R_2 with

$$\mathbb{P}(R_1 = 100) = \frac{1}{2}, \qquad \mathbb{P}(R_1 = -50) = \frac{1}{2},$$
$$\mathbb{P}(R_2 = 150) = \frac{1}{3}, \qquad \mathbb{P}(R_2 = -60) = \frac{2}{3}.$$

Which one you prefer?

$$\mathbb{E}R_1 = \frac{1}{2}100 + \frac{1}{2}(-50) = 25,$$
$$\mathbb{E}R_2 = \frac{1}{3}150 + \frac{2}{3}(-60) = 10.$$

So R_1 seems to be better.

* St. Petersburg Paradox: (posed by Nicolas Bernoulli in 1713)

$$\mathbb{P}(X = 2^n) = \frac{1}{2^n}, \qquad n \geq 1,$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} 2^n \mathbb{P}(X = 2^n) = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \infty.$$

Question: How much are you willing to pay to play the game?

How about \$10,000? Or \$1,000? Or ???

In 1738, Daniel Bernoulli (a cousin of Nicolas) introduced **expected utility**: $\mathbb{E}[u(X)]$. With $u(x) = \sqrt{x}$, one has

$$\mathbb{E}\sqrt{X} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = 1 + \sqrt{2}.$$

* 1944, von Neumann–Morgenstern: Introduced "rationality" axioms: Completeness, Transitivity, Independence, Continuity.

Standard stochastic optimal control theory is based on the expected utility theory.

- Decision-making based on expected utility theory is **time-consistent**.
- In classical expected utility theory, the probability is **objective**.
- It is controversial whether a probability should be **objective**.
- Early relevant works: Ramsey (1926), de Finetti (1937)

Allais Paradox (1953). Let X be a payoff

Option 1. $\mathbb{P}(X_1 = 100) = 100\%$

Option 2. $\mathbb{P}(X_2 = 100) = 89\%$, $\mathbb{P}(X_2 = 0) = 1\%$,
$\mathbb{P}(X_2 = 500) = 10\%$

Option 3. $\mathbb{P}(X_3 = 0) = 89\%,\ \mathbb{P}(X_3 = 100) = 11\%$

Option 4. $\mathbb{P}(X_4 = 0) = 90\%,\ \mathbb{P}(X_4 = 500) = 10\%$

Most people have the following preferences:

$$X_2 \prec X_1, \qquad X_3 \prec X_4.$$

If there exists a utility function $u : \mathbb{R} \to \mathbb{R}^+$ such that

$$X \prec Y \quad \Longleftrightarrow \quad \mathbb{E}[u(X)] < \mathbb{E}[u(Y)],$$

then

$$\begin{aligned} X_2 \prec X_1 \quad \Rightarrow \quad & \mathbb{E}[u(X_2)] = 0.89u(100) + 0.1u(500) + 0.01u(0) \\ & {\color{red}<} \mathbb{E}[u(X_1)] = u(100) \\ X_3 \prec X_4 \quad \Rightarrow \quad & \mathbb{E}[u(X_3)] = 0.89u(0) + 0.11u(100) \\ & {\color{blue}<} \mathbb{E}[u(X_4)] = 0.9u(0) + 0.1u(500), \end{aligned}$$

Thus,

$$0.11u(100) {\color{red}>} 0.1u(500) + 0.01u(0),$$

$$0.11u(100) {\color{blue}<} 0.01u(0) + 0.1u(500).$$

Relevant Literature:

* Subjective expected utility theory (Savage 1954)

* Mean-variance preference (Markowitz 1952)
  leading to nonlinear appearance of conditional expectation

* Choquet integral (1953)
  leading to Choquet expected utility theory

* Prospect Theory (Kahneman–Tversky 1979)
  (Kahneman won 2002 Nobel Prize)

* Distorted probability (Wang–Young–Panjer 1997)
  widely used in insurance/actuarial science

* BSDEs, g-expectation (Peng 1997)
  leading to time-consistent nonlinear expectation

* BSVIEs (Yong 2006,2008)
  leading to time-inconsistent dynamic risk measure

Recent Relevant Literatures:

* Björk–Murgoci (2008), Björk–Murgoci–Zhou (2013)

* Hu–Jin–Zhou (2012, 2015)

* Yong (2014, 2015)

* Björk–Khapko–Murgoci (2016)

* Hu–Huang–Li (2017)

- **A Summary:**

Time-Preferences: (Exponential/General) Discounting.

Risk-Preferences: (Subjective/Objective) Expected Utility.

Exponential discounting + **objective** expected utility/disutility leads to **time-consistency**.

Otherwise, the problem will be **time-inconsistent**.

3. Equilibrium Strategies

Time-consistent solution:

Instead of finding an optimal solution
(which is **time-inconsistent**),

find an equilibrium strategy
(which is **time-consistent**).

Sacrifice some immediate satisfaction,

save some for the future

(retirement plan, controlling economy growth speed, ...)

A General Formulation:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

with

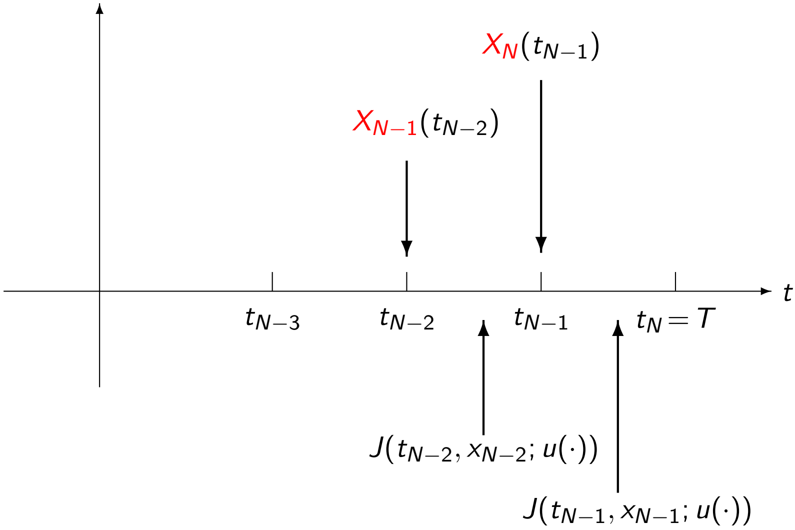
$$J(t, x; u(\cdot)) = \mathbb{E}_t\Big[\int_t^T g(t, s, X(s), u(s))ds + h(t, X(T))\Big].$$

$$\mathcal{U}[t, T] = \Big\{ u : [t, T] \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \Big\}.$$

Problem (N). For given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)).$$

This problem is **time-inconsistent**.


$X_N(t_{N-1})$
$X_{N-1}(t_{N-2})$
t
t_{N-3}
t_{N-2}
t_{N-1}
$t_N = T$
$J(t_{N-2}, x_{N-2}; u(\cdot))$
$J(t_{N-1}, x_{N-1}; u(\cdot))$

Idea of Seeking Equilibrium Strategies.

• Partition the interval $[0, T]$:

$$[0, T] = \bigcup_{k=1}^{N} [t_{k-1}, t_k], \qquad \Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N.$$

• Solve an optimal control problem on $[t_{N-1}, t_N]$, with cost functional:

$$J_N(u) = \mathbb{E}\Big[h(t_{N-1}, X(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-1}, s, X(s), u(s))ds\Big],$$

obtaining optimal pair $(X_N(\cdot), u_N(\cdot))$, depending on the initial pair (t_{N-1}, x_{N-1}).

• Solve an optimal control problem on $[t_{N-2}, t_{N-1}]$ with a **sophisticated** cost functional:

$$J_{N-1}(u) = \mathbb{E}\Big[h(t_{N-2}, X(T)) + \int_{t_{N-1}}^{t_N} g(t_{N-2}, s, X_N(s), u_N(s))ds + \int_{t_{N-2}}^{t_{N-1}} g(t_{N-2}, s, X(s), u(s))ds\Big].$$

• By induction to get an **approximate equilibrium strategy**, depending on Π.

• Let $\|\Pi\| \to 0$ to get a limit.

Definition. $\Psi : [0,T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* if for any $x \in \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0,T], \\ \bar{X}(0) = x \end{cases}$$

admits a unique solution $\bar{X}(\cdot)$. For some $\Psi^\Pi : [0,T] \times \mathbb{R}^n \to U$,

$$\lim_{\|\Pi\| \to 0} d\Big(\Psi^\Pi(t,x), \Psi(t,x)\Big) = 0,$$

uniformly for (t,x) in any compact sets, where
$\Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, and

$$\begin{aligned} & J^k\big(t_{k-1}, X^\Pi(t_{k-1}); \Psi^\Pi(\cdot)\big|_{[t_{k-1},T]}\big) \\ & \leq J^k\big(t_{k-1}, X^\Pi(t_{k-1}); u^k(\cdot) \oplus \Psi^\Pi(\cdot)\big|_{[t_k,T]}\big), \quad \forall u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k], \end{aligned}$$

$J^k(\cdot)$ — **sophisticated** cost functional.

$$
\begin{cases}
dX^{\Pi}(s) = b(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))ds \\
\qquad\qquad +\sigma(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))dW(s), \quad s \in [0, T], \\
X^{\Pi}(0) = x
\end{cases}
$$

$$
[u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k, T]}](s) = \begin{cases} u^k(s), & s \in [t_{k-1}, t_k), \\ \Psi^{\Pi}(s, X^k(s)), & s \in [t_k, T], \end{cases}
$$

$$
\begin{cases}
dX^k(s) = b(s, X^k(s), u^k(s))ds \\
\qquad\qquad +\sigma(s, X^k(s), u^k(s))dW(s), \quad s \in [t_{k-1}, t_k), \\
dX^k(s) = b(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))ds \\
\qquad\qquad +\sigma(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))dW(s), \quad s \in [t_k, T], \\
X^k(t_{k-1}) = X^{\Pi}(t_{k-1}).
\end{cases}
$$

Equilibrium control:

$$\bar{u}(s) = \Psi(s, \bar{X}(s)), \qquad s \in [0, T].$$

Equilibrium state process $\bar{X}(\cdot)$, satisfying:

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0, T], \\ \bar{X}(0) = x \end{cases}$$

Equilibrium value function:

$$V(t, \bar{X}(t)) = J(t, \bar{X}(t); \bar{u}(\cdot)).$$

The previous explained idea will help us to get such a $\Psi(\cdot\,,\cdot)$.

Let $D[0,T]=\{(\tau,t)\mid 0\le\tau\le t\le T\}$. Define

$$a(t,x,u)=\frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T,\quad \forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U,$$

$$\mathbb{H}(\tau,t,x,u,p,P)=\mathrm{tr}\left[a(t,x,u)P\right]+\langle\, b(t,x,u),p\,\rangle+g(\tau,t,x,u),$$
$$\forall(\tau,t,x,u,p,P)\in D[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{S}^n,$$

Let $\psi:\mathcal{D}(\psi)\subseteq D[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n\to U$ such that

$$\mathbb{H}(\tau,t,x,\psi(\tau,t,x,p,P),p,P)=\inf_{u\in U}\mathbb{H}(\tau,t,x,u,p,P)>-\infty,$$
$$\forall(\tau,t,x,p,P)\in\mathcal{D}(\psi).$$

In **classical** case, it just needs

$$H(t,x,p,P)=\inf_{u\in U}\mathbb{H}(t,x,u,p,P)>-\infty,$$
$$\forall(t,x,p,P)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n.$$

Equilibrium HJB equation:

$$\begin{cases}\Theta_t(\tau,t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\Theta_{xx}(\tau,t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle\\ +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(\tau,t,x))\big)=0,\ \ (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}$$

Classical HJB Equation:

$$\begin{cases}\Theta_t(t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)\Theta_{xx}(t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big),\Theta_x(t,x)\,\rangle\\ +g\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)=0,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

or

$$\begin{cases}\Theta_t(t,x)+H\big(t,x,\Theta_x(t,x),\Theta_{xx}(t,x)\big)=0,\quad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

Equilibrium value function:

$$V(t,x)=\Theta(t,t,x), \qquad \forall(t,x)\in[0,T]\times\mathbb{R}^n.$$

It satisfies

$$V(t,\bar X(t;x))=J\big(t,\bar X(t;x);\Psi(\cdot)\big|_{[t,T]}\big), \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

Equilibrium strategy:

$$\Psi(t,x)=\psi(t,t,x,V_x(t,x),V_{xx}(t,x)), \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

Theorem. *Under proper conditions, the equilibrium HJB equation admits a unique classical solution $\Theta(\cdot\,,\cdot\,,\cdot)$. Hence, an equilibrium strategy $\Psi(\cdot\,,\cdot)$ exists.*

Equilibrium strategy $\Psi(\cdot\,,\cdot)$ has the following properties:

- **Time-consistent:** $t \mapsto \Psi(t, \bar{X}(t))$.
- **Local approximately optimality:**

For any $t \in [0, T)$, any $\varepsilon > 0$, and any $u(\cdot) \in \mathcal{U}[t, t+\varepsilon)$, let

$$\big[u(\cdot) \oplus \Psi(\cdot\,,\cdot)\big](s,x) = \begin{cases} u(s), & (s,x) \in [t, t+\varepsilon) \times \mathbb{R}^n, \\ \Psi(s,x), & (s,x) \in [t+\varepsilon, T] \times \mathbb{R}^n. \end{cases}$$

The following holds:

$$J\big(t, \bar{X}(t); \bar{\Psi}(\cdot\,, \bar{X}(\cdot))\big) \le J\big(t, x; u(\cdot) \oplus \Psi(\cdot\,, \bar{X}(\cdot))\big) + o(\varepsilon).$$

(Perturbed on $[t, t+\varepsilon)$.)

Return to Generalized Merton Problem

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s)dW(s), \\ X(t) = x. \end{cases}$$

$$J(t, x; u(\cdot), c(\cdot)) = \mathbb{E}_t\Big[\int_t^T \nu(t, s)c(s)^\beta(s)ds + \rho(t)X(T)^\beta\Big],$$

with $\beta \in (0, 1)$. Classical case:

$$\nu(t, s) = e^{-\delta(s-t)}, \quad \rho(t) = e^{-\delta(T-t)}, \qquad 0 \le t \le s \le T.$$

Problem. Find $(\bar{u}(\cdot), \bar{c}(\cdot))$ to maximize $J(t, x; u(\cdot), c(\cdot))$.

Time-consistent equilibrium strategy:

$$\Psi(t,x) = \varphi(t)x^{\beta},$$

with $\varphi(\cdot)$ satisfying integral equation:

$$\varphi(t) = e^{\lambda(T-t)-\beta\int_t^T (\frac{\nu(s',s')}{\varphi(s')})^{\frac{1}{1-\beta}}ds'}\rho(t) + \int_t^T e^{\lambda(s-t)-\beta\int_t^s (\frac{\nu(s',s')}{\varphi(s')})^{\frac{1}{1-\beta}}ds'}\Big(\frac{\nu(s,s)}{\varphi(s)}\Big)^{\frac{\beta}{1-\beta}}\nu(t,s)ds,$$

$$t \in [0, T].$$

4. Open Problems

1. The well-posedness of the equilibrium HJB equation for the case $\sigma(t, x, u)$ is **not independent** of u.
2. The case that ψ is **not unique**, has **discontinuity**, etc.
3. The case that $\sigma(t, x, u)$ is **degenerate**, viscosity solution?
4. **Random** coefficient case (non-degenerate/degenerate cases).
5. The case involving **conditional expectation**.
6. **Infinite horizon** problems.

Thank You!