Optimal Investment, Indifference Pricing and Dynamic Default Insurance in the Presence of Defaults

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Research Goals

Solve the optimal investment problem when the underlying traded asset may default.

- Price defaultable bonds.
- Price dynamic default insurance.

Obtain explicit answers.

- Provide a PDE counterpart to the BSDE pricing literature.

Motivation

Say our goal is to price a claim whose payoff is contingent upon survival of a reference entity.

- Payoff: $\phi_T 1_{T>\delta}$
- δ: default time of a firm S.

In practice, pricing is done under a risk neutral measure.

Two problems:

- What risk neutral measure?
- What is the underlying traded asset? What if the underlying is the reference entity?

Motivation

Say our goal is to insure ourselves against losses from the default of a stock in which we own a position.

We could enter into a CDS

- What if investment horizon does not match CDS maturity?
- What if we want dynamic protection?

Is there a fair price for dynamic protection taking into account market incompleteness, and our preferences?

Contribution to the Literature

Optimal investment and indifference pricing with defaults have been extensively studied.

- Primarily from the "BSDE" perspective, especially with respect to pricing.
- We fill in a gap by considering Markovian factor models, using PDE techniques, and focusing on indifference pricing.
  - Amenable to computation and analysis.

The computation of dynamic default insurance has been much less well-studied.

Contribution to the Literature

(selected) "PDE" articles

- [Lin06]: Merton model with default intensity $\gamma_t = \gamma(S_t)$ under a fixed risk neutral measure. Analytical formulas for European option prices.
- [SZ07]: single stock factor model similar to ours. However, investor does not lose money in stock upon default.
- [BBC16]: risk-sensitive control problem in factor model with multiple securities, default state dependent intensities. Investor does not lose money in stock upon default.
- [BC16]: optimal investment/consumption problem for power utility in a factor model with multiple securities, default state dependent intensities. Investor loses money upon default.

Contribution to the Literature

(selected) "BSDE" articles

- [Mor09, LQ11]: single stock and non-traded claim. Brownian setting prior to default.
- [JP11, JKP13]: single/multiple stocks along with claim. Multiple credit events which cause a jump in stock prices with trading possible after jump. Brownian setting
- [MS17]: stock modeled as a pure-jump Levy process.
- [LQ15]: extension of [LQ11] to partial information models.
- [GN15, CGN15]: mean-variance hedging under default risk.

Model

Reduced form, "hybrid" intensity model: [SZ07].

X: underlying factor process

- $dX_t = b(X_t)dt + a(X_t)dW_t$.
  - W: d-dim B.M.. $b, A := aa'$ smooth, A locally elliptic.
- Solution to Martingale problem for L on $E \subseteq \mathbb{R}^d$ where
  - $L = (1/2)\mathrm{Tr}(AD^2) + b'\nabla$
  - $E = \cup_n E_n$ with E_n bounded, $E_n \uparrow$, ∂E_n smooth.

One risky asset S (riskless asset set to 1)

- S defaults at the random time δ. Prior to δ, S has instantaneous returns, variances, correlations driven by X.

Model

Start at $t \geq 0$. $X_t^{t,x} = x \in E$. Write $X = X^{t,x}$.

$$\frac{dS_s}{S_s} = 1_{s \leq \delta} \left((\mu - \gamma)(X_s) ds + (\sigma\rho)(X_s)' dW_s + \left(\sigma\sqrt{1 - \rho'\rho}\right)(X_s) dW_s^0 \right) \\ - dM_s; \qquad s \geq t.$$

- W^0: one-dim B.M. $\perp\!\!\!\perp$ of W.
- $\delta := \inf \left\{ s > t : \int_t^s \gamma(X_u) du = -\log(U) \right\}$, $U \perp\!\!\!\perp W, W^0$.
- $H_s := 1_{s \geq \delta}$; $\qquad M_s := H_s - \int_t^{s \wedge \delta} \gamma(X_u) du$,
- $\mathbb{G} := \mathbb{F}^{W, W^0} \vee \mathbb{F}^H$. W, W^0, M are $\mathbb{G}$ local martingales.
- $\mu, \sigma, \gamma, \rho$ smooth functions on E, $\gamma, \sigma > 0$, $\rho'\rho \leq 1$.

Optimal Investment Problem

Investment horizon: $[t, T]$ for $T > t$.

$\mathcal{M}$: equivalent local martingale measures on $\mathcal{G}_T$. $\widetilde{\mathcal{M}}$ subset with finite relative entropy w.r.t. $\mathbb{P}$.

$\mathcal{A}$: acceptable (dollar) trading strategies π.

- Wealth process $\mathcal{W}_\cdot^{\pi,w} = w + \int_t^\cdot \pi_u dS_u / S_{u-}$.
- Dollar position π_δ lost at δ.

$\pi \in \mathcal{A}$ if $\mathcal{W}^{\pi,w}$ is a $\mathbb{Q}$ local martingale for all $\mathbb{Q} \in \widetilde{\mathcal{M}}$.

Optimal Investment Problem

Exponential investor: $U(w) := -e^{-\alpha w}, w \in \mathbb{R}$.

Investor

- Trades in S according to $\pi \in \mathcal{A}$.
- Owns a non-traded claim with time T payoff $\phi(X_T)1_{T<\delta}$.
  - ϕ smooth, bounded. Primarily care about $\phi \equiv 1, \phi \equiv 0$.

For 0 initial wealth write $\mathcal{W}^\pi = \mathcal{W}^{\pi,0}$ and define

$$u(t,x;\phi) := \sup_{\pi \in \mathcal{A}} E\left[-e^{-\alpha(\mathcal{W}_T^\pi + \phi(X_T)1_{\delta > T})}\right]; \quad (X_t = x)$$

$$G(t,x;\phi) := -\frac{1}{\alpha}\log\left(-u(t,x;\phi)\right).$$

$G(t,x;\phi) = -\frac{1}{\alpha}\log\left(-u(t,x;\phi)\right)$: Certainty Equivalent

Heuristics using DPP suggest G should solve

$$0 = G_t + LG - \frac{\alpha}{2}\nabla G' A \nabla G + \frac{\sigma^2}{2\alpha}\left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho\right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G\right);$$

$$\phi = G(T,\cdot)$$

- $\theta(y)$: inverse of ye^y and $\theta_G := \theta\left(\frac{\gamma}{\sigma^2}e^{\frac{\mu}{\sigma^2}+\alpha G-\frac{\alpha}{\sigma}\nabla G' a\rho}\right)$.

If G is a classical solution, DPP suggests optimal strategy is

- $\hat{\pi}_s = \hat{\pi}(s, X_s^{t,x})$ for $\hat{\pi} = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho - \theta_G\right)$.

Certainty Equivalent PDE

$$0 = G_t + LG - \frac{\alpha}{2}\nabla G' A \nabla G + \frac{\sigma^2}{2\alpha}\left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho\right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G\right);$$
$$\phi = G(T, \cdot)$$

- This is a semi-linear degenerate parabolic PDE.
  - Non-linearities arise due to market incompleteness.
- Luckily: $\theta(y) \approx \log(y) - \log(\log(y)), y >> 0$.
  - PDE is quadratically growing in $G, \nabla G$.
- Regarding solutions/verification:
  - For general regions E, local ellipticity, verification is hard: lack gradient estimates near ∂E.
- We must enforce some additional (global) condition.

The Main Assumption

Set $\ell := (\mu - \gamma)/\sigma$ (market price of risk).

Today: assume "strictly incomplete" market absent default.

- The paper treats the "complete" case as well.

Main assumptions:

- $\sup_{x\in E} \rho'\rho(x) < 1$.
- For some $\varepsilon > 0$ we have for each n

$$\sup_{x\in \overline{E}_n} E^x\left[e^{\varepsilon\int_0^T \ell(X_u)^2 du}\right] = C(\varepsilon, n) < \infty.$$

This assumption is MILD. Holds in virtually all models.

- E.g. $X \sim OU, CIR$, μ, σ^2, γ affine.

The Main Result

$$0 = G_t + LG - \frac{\alpha}{2}\nabla G' A \nabla G + \frac{\sigma^2}{2\alpha}\left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho\right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G\right);$$

$$\phi = G(T, \cdot)$$

<u>Theorem</u>: assume $\sup_{x\in E} \rho'\rho(x) < 1$ and for some $\varepsilon > 0$:

- $\sup_{x\in\overline{E}_n} E^x\left[e^{\varepsilon\int_0^T \ell(X_u)^2 du}\right] = C(n) < \infty$, $\forall n$.

Then

- The certainty equivalent G is a classical ($C^{1,2}$) solution.
- The optimal trading strategy is
  - $\hat{\pi}_s = \hat{\pi}(s, X_s^{t,x})$ for $\hat{\pi} = \frac{1}{\alpha}\left(\frac{\mu}{\sigma} - \frac{\alpha}{\sigma}\nabla G' a\rho - \theta_G\right)$.
- The optimal martingale measure $\hat{\mathbb{Q}}$ has density
  - $\hat{Z}_s = e^{-\alpha\left(\mathcal{W}_s^{\hat{\pi}} - G(t,x;\phi) + 1_{\delta>s}G(x,X_s;\phi)\right)}$.

Application: Pricing for Defaultable Bonds

Investor owns q units notional: claim payoff $q1_{\delta > T}$.

(per-unit, buyer's) indifference price: $p(t,x;q)$ solving

- $u(t,x;0,0) = u(t,x;q,-qp(t,x;q)) = e^{\alpha qp(t,x,;q)}u(t,x;q,0)$.
  - $u(t,x;\phi,w)$: utility for initial wealth w.
  - Well known p does not depend on w.

Immediate result as $G(t,x;q) = -(1/\alpha)\log(-u(t,x;q))$:

- $p(t,x;q) = \frac{1}{q}\left(G(t,x;q) - G(t,x;0)\right)$.

Application: Dynamic Default Insurance

Goal: find a fair price for dynamic protection against default.

- Approximation to CDS pricing valid for frequent contract adjustments.

Motivation from [SZ07]: optimal investment/pricing but with no loss at default.

- π_δ not lost at default time δ.

How is this possible? What contact has been entered into which enables this?

Dynamic Default Insurance

Perspective: investor has two alternatives:

- A) Do not purchase protection. Lose π_δ at δ. Indirect utility of $u(t,x)$.
- B) Purchase protection. Pay a (per-unit) cash flow rate of f, where f is to-be-determined.
  - Wealth dynamics:

$$
\begin{aligned}
d\mathcal{W}_s^{\pi,d} &= \pi_s 1_{s\le\delta}\left((\mu-\gamma)(X_s) - f_s\right) ds \\
&\quad + \pi_s 1_{s\le\delta}\left((\sigma\rho)(X_s)' dW_s + (\sigma\sqrt{1-\rho'\rho})(X_s) dW_s^0\right).
\end{aligned}
$$

  - Indirect utility

$$
u^d(t,x) := \sup_{\pi\in\mathcal{A}_d} E\left[-e^{-\alpha\mathcal{W}_T^{\pi,d}}\right].
$$

Dynamic Default Insurance

$$G(t,x) = -\frac{1}{\alpha}\log\left(-u(t,x)\right); \quad G^d(t,x) = -\frac{1}{\alpha}\log\left(-u^d(t,x)\right).$$

Guess $f_t = f(t, X_t)$. Find f so that PDEs for G, G^d are the same (both have terminal condition ϕ).

$$\begin{aligned}
0 &= G_t + LG - \frac{\alpha}{2}\nabla G' A \nabla G \\
&\quad + \frac{\sigma^2}{2\alpha}\left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho\right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G\right); \\
0 &= G_t^d + LG^d - \frac{\alpha}{2}\nabla (G^d)' A \nabla G^d \\
&\quad + \frac{\sigma^2}{2\alpha}\left(\left(\frac{\mu - f}{\sigma^2} - \frac{\alpha}{\sigma}\nabla (G^d)' a\rho\right)^2 + \frac{2\gamma}{\sigma^2}\left(1 - e^{\alpha G^d}\right)\right).
\end{aligned}$$

Dynamic Default Insurance

Upon inspection, given a solution G to the first PDE, G will solve the second PDE if f satisfies

$$\frac{f_\pm}{\sigma^2} = \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}\nabla G' a\rho\right)^2 - \left(\theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2}e^{\alpha G}\right)}.$$

- Term inside square root is non-negative: real solutions.
- We choose the "-" solution.
  - Lowest possible f since this is what the investor pays.
  - Can also justify f_- by inspecting optimal strategies $\pi^d_\pm$: $f_+ > 0$ and $\pi^d_+ < 0$ - not feasible.

Dynamic Default Insurance

We define the dynamic default insurance protection price

$$f := \sigma^2 \left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 - \left(\theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G} \right)} \right).$$

Facts

- $f \leq \gamma e^{\alpha(G+\hat{\pi})} = \gamma^{\hat{\mathbb{Q}}}$: the default intensity under the dual optimal measure $\hat{\mathbb{Q}}$.
  - Equality only when $\hat{\pi} = 0$.
- $f > 0$ when $\hat{\pi} > 0$: intuitive. Pay for protection when long.
- $f > 0$ possible even when $\hat{\pi} < 0$, but $f < 0$ for $\hat{\pi} << 0$.

Numerical Application

Application: $X \sim CIR$, affine market price of risk.

- $dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}dW_t$.
- Prior to default
  - $dS_t/S_t = \mu X_t dt + \sigma\sqrt{X_t}\left(\rho dW_t + \sqrt{1-\rho^2}dW_t^0\right)$.
- Default intensity: $\gamma_t = \gamma X_t$.

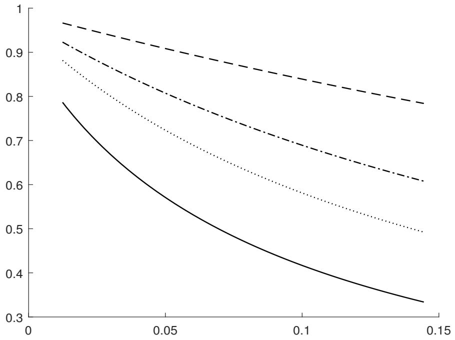
Assume $\mu \in \mathbb{R}, \sigma, \gamma > 0$ and $|\rho| < 1$.

- Main assumption holds provided $\kappa\theta > \xi^2/2$.

Application: Defaultable Bond Pricing

Investor owns q units of a defaultable bond.

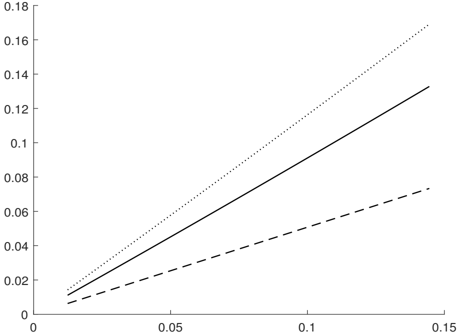
$p(0, x; q)$ as a function of q, x for $T = 1$.



- Physical default prob of 3% at $x = 6\%$ (long run mean).
- $q = 1$(dash), $q = 3$ (dot-dash), $q = 5$ (dot), $q = 10$ (solid).

Application: Dynamic Default Insurance

$f(0, x)$ as a function of x for $T = 1$.



- $\gamma^{\hat{\mathbb{Q}}}$ (dash), f (solid), γ (dash).

THANK YOU!

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