Particle Systems with Singular Interaction through Hitting Times: application in Systemic Risk modeling

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Non-core liabilities

- **Core liabilities** are defined as deposits to financial institutions (banks) from households or non-financial institutions. The rest is **non-core** (e.g. derivatives backed by the bank, deposits from other banks).
- The average level of non-core liabilities can be viewed as a measure of **connectivity** of the interbank system.
- On the one hand, **high connectivity** allows banks to raise capital when they need it.
- On the other hand, it serves as a channel for **default contagion**.
- Some economists have proposed to **control** the **non-core exposure** due to its **procyclical** nature.

Controlling non-core liabilities

- *Shin (2010)*: "The Obama administration has proposed a tax of 15 basis points (0.15%) on the non-deposit liabilities of leveraged financial institutions in the United States with assets of more than 50 billion dollars."
- *Shin (2012)*: "Beginning in June 2010, the authorities in Korea introduced ... the levy on the non-core liabilities of the banks ...".

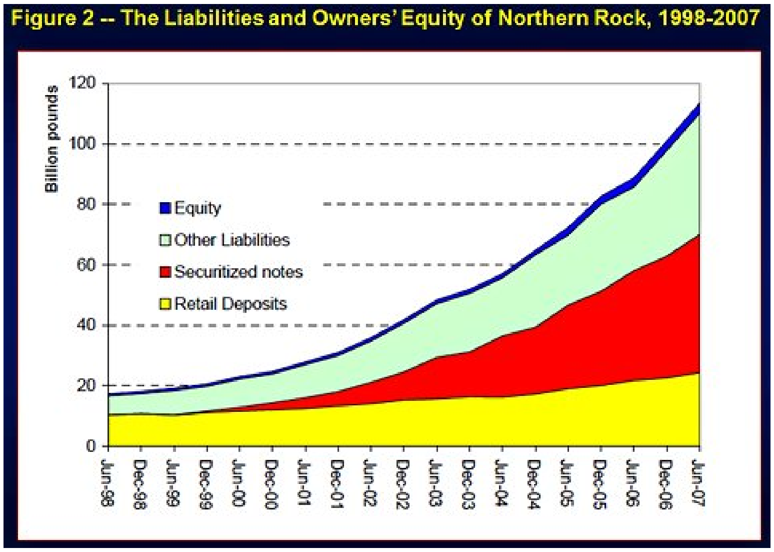

Figure: Liabilities of Northern Rock

Key features of the model

- **Dynamic**.
- **Structural**.
- Yields **endogenous** definition of **systemic event** as a phase transition: i.e. a sharp increase in the number of defaults.
- The **time** of a **systemic event** is expressed explicitly in terms of the (controllable) **level of non-core exposure** and the (observable) fraction of firms that are **about to default**.
- The **mathematical model** is similar to the one considered in *Delarue et al. (2015)*, with application in Neuroscience. It is **interesting in its own right**.

Finite particle system

- The i-th firm (bank) defaults when its (normalized) **value** X^i hits the **default barrier** 1. Denote $Y^i = \log X^i$.
- Assume:

$$Y^i_t = Y^i_0 + \alpha\, dt + \sigma\, B^i_t + C\, \log\left(\frac{1}{N}\, S_t\right),$$

$$S_t = \sum_{j=1}^{N} \mathbf{1}_{\{\tau^j > t\}}, \quad \tau^i := \inf\{t :\; Y^i_t \leq 0\}, \quad i = 1, 2, \ldots, N.$$

- Rationale: if k firms default at time t, the value of each other bank is multiplied by $(1 - k/S_{t-})^C \approx 1 - Ck/S_{t-}$.
- Default **cascade** size:

$$D_t = \inf\left\{k = 1, 2, \ldots, S_{t-} \;:\; Y^{(k)}_{t-} + C\log\left(1 - \frac{k-1}{S_{t-}}\right) > 0\right\} - 1.$$

- Note: $\sum_{u \leq t:\, D_u > 0} C\log\left(1 - \frac{D_u}{S_{u-}}\right) = C\log\left(\frac{1}{N}\, S_t\right)$.

Large-population limit

- **Goal**: analyze the limit of **empirical measure** $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y^i}$, as $N \to \infty$.
- **Definition.** $(\overline{Y}_t)$ is a **physical solution** if
  - $\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t,$
    $\overline{\tau} := \inf\{t : \overline{Y}_t \leq 0\}, \quad \Lambda_t = C \log \mathbb{P}(\overline{\tau} > t),$
  - and, whenever $\Lambda_t \neq \Lambda_{t-}$, we have $\Lambda_{t-} - \Lambda_t \leq F_t(\overline{D}_t)$, where
    $F_t(y) = -C \log\left(1 - \mathbb{P}(\overline{Y}_{t-} \in (0,y)\,|\,\overline{\tau} \geq t)\right) > 0,$
    $\overline{D}_t := \inf\{y > 0 : y > F_t(y)\}.$
- The time of the first **systemic event** is $t_{sys} = \inf\{t : \Lambda_t \neq \Lambda_{t-}\}$.
- For $y \approx 0$, $F_t(y) \approx Cp(t,0)y$, where $p(t,\cdot)$ is the density of $(Y_{t-}\,|\,\overline{\tau} \geq t)$.
- Then, $t_{sys} \geq \inf\{t : p(t,0) > 1/C\}$.

Jumps of Λ

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t, \quad \inf\{t\,:\,\Lambda_t \neq \Lambda_{t-}\} = t_{sys} \geq \inf\{t\,:\,p(t,0) > 1/C\},$$

$p(t,\cdot)$ is the density of $(Y_{t-}\,|\,\overline{\tau} \geq t)$.

- **Theorem.** There exists a constant $c^* = c^*(\sigma) < \infty$ such that, for any physical solution $\overline{Y}$, with $\overline{Y}_0$ admitting a density, we have $\Lambda_t \neq \Lambda_{t-}$ whenever
$$\lim_{\varepsilon \downarrow 0} \operatorname{ess\,inf}_{y \in (0,\varepsilon)} p(t,y) > c^*/C$$

- Heuristically, we obtain
$$t_{sys} \leq \inf\{t\,:\,p(t,0) > c^*/C\}$$

- **Proof** is based on stochastic dominance: construct a simpler process that coincides with $\overline{Y}_{t-}$ before t and dominates $\overline{Y}$ from above; prove that it jumps down at t.

Convergence

$$Y^i_t = Y^i_0 + \alpha\, dt + \sigma\, B^i_t + C\,\log\left(\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\{\tau^j > t\}}\right), \quad \mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y^i}$$

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + C\log \mathbb{P}(\overline{\tau} > t).$$

Theorem. Assume that $\{Y^i_0\}$ are i.i.d. with a bounded density f vanishing in a neighborhood of 0. Then,

- the sequence of random measures $\{\mu^N \in \mathcal{P}(\mathcal{P}(D))\}$ is **tight** with respect to the topology of weak convergence induced by the **Skorokhod M1 topology**,
- and **every limit point** of this sequence belongs with probability one to the space of distributions of **physical solutions** $\overline{Y}$ with $\overline{Y}_0 \overset{d}{=} f(y)dy$.

Sketch of the proof

$$Y_t^i = Y_0^i + \alpha\, dt + \sigma\, B_t^i + C\, \log\left(\frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{\tau^j > t\}} \right),$$

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{Y^i}, \quad \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{\{\tau^j > t\}} = \langle \mu^N, \mathbf{1}_{\{\inf_{s\in[0,t]} Y_s > 0\}} \rangle$$

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + C \log \mathbb{E}(\mathbf{1}_{\{\inf_{s\in[0,t]} \overline{Y}_s > 0\}}).$$

Proof is based on *Delarue et al. (2015)*.

- **Compactness**.
  - **Issue**: no control over the **jumps** of $\frac{1}{N} S_t = \langle \mu^N, \mathbf{1}_{\{\inf_{s\in[0,t]} Y_s > 0\}} \rangle$.
  - **Solution**: **M1 topology**.
- **Continuity**.
  - **Issue**: the mapping $Y \mapsto \mathbf{1}_{\{\inf_{s\in[0,t]} Y_s > 0\}} \rangle$ is **discontinuous**.
  - **Solution**: ensure that the paths are **noisy** and prove continuity on this set.

Uniqueness

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t, \quad \Lambda_t = C \log \mathbb{P}(\overline{\tau} > t),$$
$$\lambda_t = \tfrac{d}{dt}\Lambda_t, \quad t_{reg} = \sup\left\{t \,:\, \|\lambda\|_{L^2([0,t])} < \infty\right\}.$$

Theorem. Assume $f \in W_2^1([0,\infty))$ and $f(0) = 0$. Then,

(a) there **exists** a physical solution $\overline{Y}$, s.t. $\overline{Y}_0 \overset{d}{=} f(y)dy$ and

$$t_{reg} > 0, \quad \lim_{t\uparrow t_{reg}} \|\lambda\|_{L^2([0,t])} = \infty;$$

(b) the value of $t_{reg} > 0$ is the **same** for all physical solutions satisfying the conditions in part (a), and they are **indistinguishable** on $[0, t_{reg})$;

(c) the **density** $p(\cdot,\cdot)$, is continuous on $[0, t_{reg}) \times [0,\infty)$, with $p(\cdot, 0) \equiv 0$; moreover,

$$\lambda_t = -C\,\frac{\sigma^2}{2}\,\partial_y p(t,0).$$

Sketch of the proof

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t, \quad \Lambda_t = C \log \mathbb{P}(\overline{\tau} > t),$$
$$\lambda_t = \tfrac{d}{dt}\Lambda_t, \quad t_{reg} = \sup\left\{t \,:\, \|\lambda\|_{L^2([0,t])} < \infty\right\}.$$

Proof consists in the **contraction** property of the mapping

$$L^2 \ni \lambda \mapsto \tilde{p} \mapsto \lambda \in L^2,$$

where $\tilde{p}(\cdot,\cdot) \in W_2^{1,2}$ solves

$$\begin{cases} \partial_t \tilde{p}(t,y) = -(\alpha + \lambda_t)\, \partial_y \tilde{p}(t,y) + \frac{\sigma^2}{2} \partial^2_{yy} \tilde{p}(t,y) \\ \tilde{p}(0,y) = f(y), \quad \tilde{p}(t,0) = 0 \end{cases}$$

and

$$\lambda_t = -C\, \frac{\sigma^2}{2}\, \frac{\partial_y \tilde{p}(t,0)}{\int_0^\infty \tilde{p}(t,y)dy}.$$

A priori regularity

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t, \quad \Lambda_t = C \log \mathbb{P}(\overline{\tau} > t),$$

- **Theorem.** Assume that $\overline{Y}_0$ has a bounded density vanishing in a neighborhood of 0. Consider any t' s.t.

$$\lim_{\eta \downarrow 0} \sup_{s \in [0,t']} \operatorname{ess\,sup}_{y \in (0,\eta)} p(s,y) = 0.$$

  Then, for any $t'' < t'$, there exist $K < \infty$ and $\gamma \in (0,1]$ such that

$$|\Lambda_t - \Lambda_s| \le K\, |t-s|^{(1+\gamma)/2}, \quad s,t \in [0,t''].$$

- **Proof**.
  - Use **stochastic dominance** to show that γ-Hölder continuity of $p(t,\cdot)$ at zero implies $(1+\gamma)/2$-Hölder continuity of Λ at t.
  - Generalize the method of *Krylov-Safonov* to show that $p(\cdot,0) \equiv 0$ implies γ-Hölder continuity of $p(t,\cdot)$ at zero, for some $\gamma > 0$.

Summary

- We have characterized the physical solutions

$$\overline{Y}_t = \overline{Y}_0 + \alpha\, t + \sigma\, \overline{B}_t + \Lambda_t, \quad \Lambda_t = C \log \mathbb{P}(\overline{\tau} > t),$$

as **large-population limits** of particle systems representing (normalized) log-values of banks (or, their "distances to default").

- We have established a connection between the (observable) **distribution of particles' values** and the **systemic events**

$$\inf\{t\,:\,p(t,0) > 1/C\} \leq t_{sys} = \inf\{t\,:\,\Lambda_t \neq \Lambda_{t-}\} \leq \inf\{t\,:\,p(t,0) > c^*/C\}$$

- **What is missing**?
  - $c^* = 1$?
  - Global **uniqueness** result? or, at least, on $[0, t_{sys})$?
  - The above may be obtained via additional **a priori regularity**.