Eigenvalue problems arising in models with small transaction costs

Ryan Hynd

University of Pennsylvania

Black-Scholes

- **Money market** with interest rate $r = 0$
- One **stock**

$$dP(t) = \alpha P(t)dt + P(t)dW(t), \quad t \in [0, T]$$

- European option paying $g(P(T))$?

$\psi(t, P(t))$

$$\boxed{\psi_t + \frac{1}{2}\sigma^2 p^2 \psi_{pp} = 0}$$

for $t < T$ and

$$\psi(T, p) = g(p)$$

"Shortcomings"

- No **transaction costs**
- No **preferences**
- No **bid-ask** spread

DPZ Model

- Incorporates a bid-ask spead
- Proportional transaction costs
- Prices set by indifference pricing

Transaction costs $= \sqrt{\epsilon}$, risk aversion $= 1/\epsilon$

$$\max\left\{-z_t - \frac{1}{2}\sigma^2 p^2 \left(z_{pp} + \frac{1}{\epsilon}(z_p - y)^2\right), |z_y| - \sqrt{\epsilon}p\right\} = 0$$

for $t < T$, $p > 0$, $y \in \mathbb{R}$ and

$$z^\epsilon(T, y, p) = g(p)$$

Barles & Soner

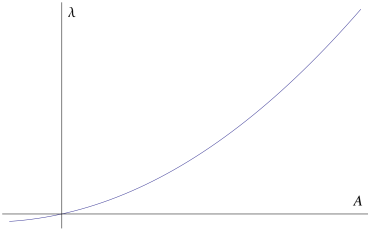
$$z^\epsilon(t,p,y) \approx \psi(t,p) + \epsilon u\left(p\frac{\psi_p(t,p)-y}{\sqrt{\epsilon}}\right)$$

Nonlinear Black-Scholes

$$\boxed{\psi_t + \lambda(p^2\psi_{pp}) = 0}$$

for $t < T$ and $p > 0$

$$\psi(T,p) = g(p)$$



"Eigenvalue" ODE

$$\max\left\{\lambda - \frac{\sigma^2}{2}(A + A^2u'' + (x+Au')^2), |u'| - 1\right\} = 0$$

Merton problem

$$v(x,y) = \sup_c \mathbb{E} \int_0^\infty e^{-\beta t} U(c(t))dt,$$

where

$$\begin{cases} dX(t) = rX(t)dt - c(t)dt \\ dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t) \\ X(t) + Y(t) \geq 0 \end{cases}$$

HJB

$$\beta v - \left(\frac{1}{2}\sigma^2 y^2 v_{yy} + \alpha y v_y + r x v_x \right) - U^*(v_x) = 0$$

$v = v(z)$ with $z = x + y \geq 0$,

$$\xi = \frac{(\alpha - r)v'(z)}{\sigma^2(-v''(z))} \quad \text{and} \quad c = U'(v'(z))$$

Davis & Norman

$$\min\left\{\beta v - \left(\frac{1}{2}\sigma^2 y^2 v_{yy} + \alpha y v_y + r x v_x\right) - U^*(v_x),\right.$$
$$\left.(1+\lambda)v_x - v_y, -(1-\mu)v_x + v_y\right\} = 0$$

where

$$x + (1-\mu)y \geq 0 \quad \text{and} \quad x + (1+\lambda)y \geq 0$$

Soner &Touzi (Wilmott & Whaley, Janecek & Shreve ...)

$\lambda = \mu = \epsilon^3$,

$$v^\epsilon(x,y) \approx v(z) - \epsilon^2 u(z) - \epsilon^4 w\left(z, \frac{y-\xi}{\epsilon}\right)$$

Asymptotics

$$v^\epsilon(x,y) \approx v(z) - \epsilon^2 u(z) - \epsilon^4 w\left(z, \frac{y-\xi}{\epsilon}\right)$$

"Eigenvalue" ODE for w

$$\max\left\{\overline{a} - \frac{1}{2}\overline{\alpha}^2 \overline{w}_{\rho\rho} - \frac{1}{2}\sigma^2\rho^2, |\overline{w}_\rho| - 1\right\} = 0$$

where

$$\overline{w}(z,\rho) = \frac{w(z,\eta\rho)}{\eta v'}, \quad \overline{\alpha} = \frac{\alpha}{\eta}, \quad \overline{a} = \frac{a}{\eta v'}$$

and $\eta = -v'/v''$.

Equation for u

$$a = \beta u - \left(\frac{1}{2}\sigma^2\xi^2 u'' + (rz + \xi(\alpha - r) - c)u'\right)$$

Question: What happens when we have n risky assets?

Option pricing in small transaction cost, large risk aversion limit

$$\max_{1\le i\le n}\left\{\lambda-\frac{1}{2}\mathsf{tr}\left[\sigma\sigma^t\left(A+AD^2uA+(x+ADu)\otimes(x+ADu)\right)\right],|u_{x_i}|-1\right\}=0$$

Portfolio optimization in small transaction cost limit

$$\max_{1\le i,j\le n}\left\{\overline{a}-\frac{1}{2}\mathsf{tr}\left[\overline{\alpha}\,\overline{\alpha}^t D_\rho^2\overline{w}\right]-\frac{1}{2}|\sigma\rho|^2,\overline{w}_{\rho_i}+\overline{w}_{\rho_j}-\lambda^{i,j}\right\}=0$$

Prototypical "eigenvalue" problem

Find $\lambda \in \mathbb{R}$ such that

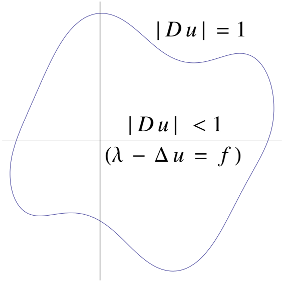
$$\max\{\lambda - \Delta u - f, |Du| - 1\} = 0, \quad x \in \mathbb{R}^n$$

has a solution $u : \mathbb{R}^n \to \mathbb{R}$.

Basic assumptions

f is **convex** and **superlinear**

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty.$$



Solution of eigenvalue problem

There is a **unique** $\lambda = \lambda^*$ such that

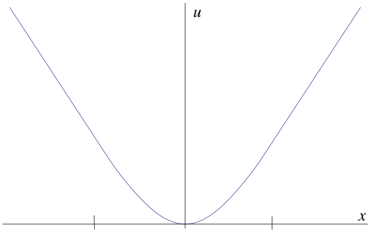
$$\max\{\lambda - \Delta u - f, |Du| - 1\} = 0$$

has a solution u

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = 1.$$

There is a convex solution u^*

$$u^*_{x_i x_j} \in L^\infty(\mathbb{R}^n).$$



Remark

If f is rotationally symmetric, $u^* \in C^2(\mathbb{R}^n)$

PDE

$$\max \{\lambda^* - \Delta u - f, |Du| - 1\} = 0$$

and $u^* : \mathbb{R}^n \to \mathbb{R}$ is a convex solution.

Three motivating questions

1. What is the "geometry" of

$$\Omega := \{x \in \mathbb{R}^n : |Du^*(x)| < 1\}?$$

2. Is u^* "unique"?
3. Regularity of u^* for problems with general convex gradient constraint?

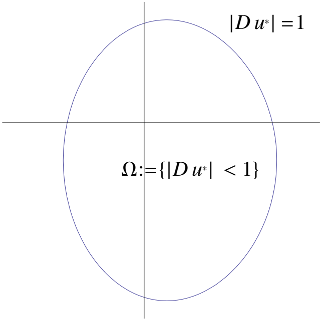
Bounded domain

As f is superlinear,

$$\lambda^* - \Delta u^* - f < 0$$

for all **large** $|x|$

$\Longrightarrow \Omega$ is **bounded**.



Conjecture

Ω is a **convex** set with **smooth** boundary.

Singular controls

$$X^\nu(t) := \sqrt{2}W(t) + \nu(t), \quad t \geq 0$$

with

$$\begin{cases} \nu(0) = 0 \\ t \mapsto \nu(t) \text{ is left-continuous} & a.s. \\ |\nu|(t) := TV_\nu[0,t) < \infty, \quad t > 0 \end{cases}$$

Ergodic problem

$$\lambda^* = \inf_\nu \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\left\{\int_0^t f(X^\nu(s))ds + |\nu|(t)\right\}$$

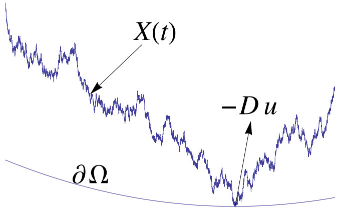
Reflected diffusion

If

$$\Omega = \{x \in \mathbb{R}^n : |Du^*(x)| < 1\}$$

has smooth boundary and $-Du^*$ is never **tangent** to $\partial\Omega$,

$$\begin{cases} dX(t) = \sqrt{2}dW(t) - Du^*(X(t))d\xi(t), t \geq 0 \\ X(0) = 0,\ \xi(0) = 0, \\ \xi(t) = \int_0^t 1_{X(s)\in\partial\Omega} d\xi(s), \quad t \geq 0 \end{cases}$$



An optimal control!

$$\nu^*(t) := -\int_0^t Du^*(X(s))d\xi(s)$$

Invariance

If u satisfies

$$\max\{\lambda^* - \Delta u - f, |Du| - 1\} = 0,$$

and

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = 1$$

then so does $u + C$.

Uniqueness known

- $n = 1$
- for rotational solutions

Like simplicity of σ_1?

$$\begin{cases} -\Delta v = \sigma_1 v, & x \in D \\ v = 0, & x \in \partial D \end{cases}$$

Basic observation

Note:

$$|Du| \leq 1 \quad \Longleftrightarrow \quad |Du|^2 \leq 1$$

so

$$\max\left\{\lambda^* - \Delta u - f, |Du|^2 - 1\right\} = 0,$$

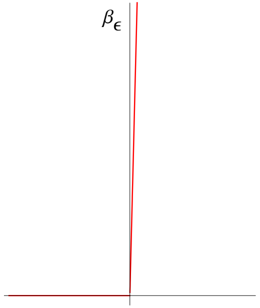
This is a **uniformly convex** gradient constraint.

Penalization

$$\lambda^* - \Delta u^\epsilon + \beta_\epsilon(|Du^\epsilon|^2 - 1) = f$$

for $x \in D$ and

$$u^\epsilon|_{\partial D} = u.$$



Smooth fit

$H : \mathbb{R}^n \to \mathbb{R}$ convex and $u : \mathbb{R}^n \to \mathbb{R}$ is a solution

$$\max \{\lambda^* - \Delta u - f, H(Du)\} = 0, \quad x \in \mathbb{R}^n.$$

Is u continuously differentiable?

Examples $(n = 3)$

- $H(Du) = u_{x_1}$
- $H(Du) = |u_{x_1}| + |u_{x_3}| - 1$
- $H(Du) = \max\{|u_{x_1}|, |u_{x_2}|, |u_{x_3}|\} - 1$

Thank You!