A Dynamic Eisenberg-Noe Model of Financial Contagion

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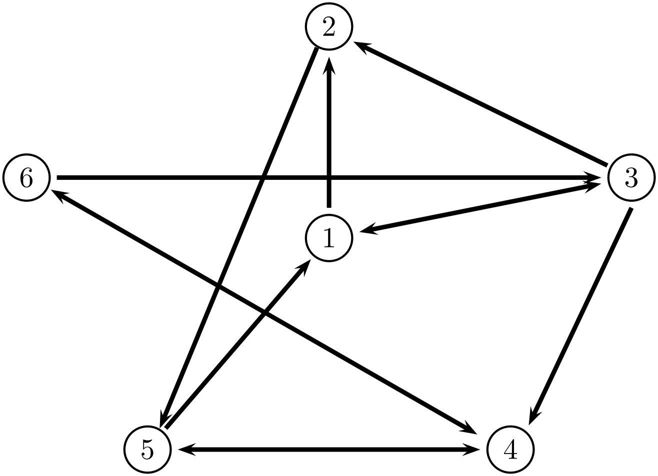
The Eisenberg & Noe Model

Network Model with Local Interactions Only:
EISENBERG & NOE (2001)

- n financial firms
- Nominal liability matrix: $(L_{ij})_{i,j=0,1,2,\ldots,n}$
- Total liabilities: $\bar{p}_i = \sum_{j=0}^{n} L_{ij}$
- Relative liabilities:

$$\pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0, \\ 0 & \text{if } \bar{p}_i = 0. \end{cases}$$

Network Model with Local Interactions Only:



Network Model with Local Interactions Only:

- Liquid endowment: $x \in \mathbb{R}^n_+$
- Obligations fulfilled via transfers of the liquid asset.
- Equilibrium computed as fixed point: $p \in \mathbb{R}^n_+$:

$$p_i = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{n} \pi_{ji} p_j \right), \quad i = 1, 2, ..., n$$

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- Existence: Tarski's fixed point theorem: maximal and minimal fixed points $p^{\downarrow} \leq p^{\uparrow}$.

Network Model with Local Interactions Only:

Uniqueness

- $S \subseteq \{1, 2, ..., n\}$ is a surplus set if $L_{ij} = 0$ and $\sum_{i \in S} x_i > 0$ for all $(i, j) \in S \times S^c$
- $o(i) = \{j \in \{1, 2, ..., n\} \mid \exists \text{ directed path from } i \text{ to } j\}$
- If $o(i)$ is a surplus set for every bank i then there exists a unique payment vector $p := p^{\uparrow} = p^{\downarrow}$ (Banach fixed point theorem)

Network Model with Local Interactions Only: Payments to Wealths

- Given clearing payments $p \in [0, \bar{p}]$ the resultant wealths are

$$V = x + \Pi^{\mathsf{T}} p - \bar{p}$$

- Given resultant wealths $V \in [x - \bar{p}, x + \Pi^{\mathsf{T}} \bar{p}]$ the clearing payments are

$$p = \left(\bar{p} - V^{-}\right)^{+}$$

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- Wealths can be computed as the fixed point of

$$V = x + \Pi^\mathsf{T} \left(\bar{p} - V^-\right)^+ - \bar{p}$$

Dynamic Eisenberg & Noe Financial Contagion Model:

Discrete Time

Discrete-Time Network Model: Capponi & Chen (2015), Ferrara, Langfield, Liu & Ota (2016), Feinstein (2017)

- Obligations owed at discrete times (e.g. clearing at the end of the day)
- Now need to distringuish between illiquidity and insolvency
- Firms can be:
  - Solvent and liquid
  - Solvent and illiquid
  - Insolvent and illiquid
  - Insolvent and liquid

Discrete-Time Network Model:

- Positive equity accrues over time
- Unpaid debts roll forward in time
- Total liabilities: $\bar{p}_i(t) = \sum_{j=0}^{n} L_{ij}(t) + V_i(t-1)^-$
- Relative liabilities: $\pi_{ij}(t) = \frac{L_{ij}(t)+\pi_{ij}(t-1)V_i(t-1)^-}{\bar{p}_i(t)}$
- Fixed point at time t:

$$V(t) = V(t-1)^+ + x(t) + \Pi(t)^\mathsf{T}\left(\bar{p}(t) - V(t)^-\right)^+ - \bar{p}(t)$$

Discrete-Time Network Model:

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Discrete-Time Network Model:

- Fixed point at time t:

$$\begin{aligned} V(t) &= V(t-1)^+ + x(t) + \Pi(t)^\mathsf{T}\left(\bar{p}(t) - V(t)^-\right)^+ - \bar{p}(t) \\ &= V(t-1) + c(t) - A(t,V)^\mathsf{T} V(t)^- + A(t-1,V)^\mathsf{T} V(t-1)^- \end{aligned}$$

$$c_i(t) = x_i(t) + \sum_{j=0}^{n} (L_{ji}(t) - L_{ij}(t))$$

$$a_{ij}(t,V) = \frac{L_{ij}(t) + a_{ij}(t-1,V)V_j(t-1)^- - \pi_{ij}(t)\left(\bar{p}_i(t) - V_i(t)^-\right)^+}{V_i(t)^-}$$

Discrete-Time Network Model:

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- Can be thought of as generalization with other choices of c_i and a_{ij} as well

Discrete-Time Network Model: Existence and Uniqueness

$$V(t) = V(t-1) + c(t) - A(t,V)^\mathsf{T} V(t)^- + A(t-1,V)^\mathsf{T} V(t-1)^-$$

- Specific model: existence and uniqueness follow exactly from EISENBERG & NOE (2001)

Discrete-Time Network Model: Existence and Uniqueness

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- Specific model: existence and uniqueness follow exactly from Eisenberg & Noe (2001)
- Generalized model: $A(t,V)$ with $A(t,V)^\mathsf{T} V^-$ bounded
  - Nonspeculative: No firm benefits from another's losses
  - Any nonspeculative system has a greatest and least fixed point $V^\uparrow(t) \geq V^\downarrow(t)$
  - If society node 0 is strictly nonspeculative and $a_{i0} > 0$ for all i, then there exists a unique fixed point $V(t)$

Discrete-Time Network Model: Example with Loans

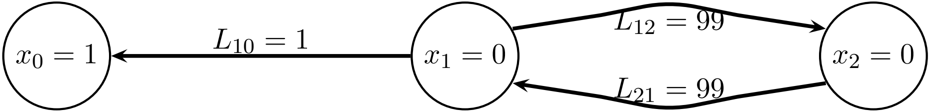
$$V(t) = V(t-1) + c(t) - A(t,V)^{\mathsf{T}} V(t)^- + A(t-1,V)^{\mathsf{T}} V(t-1)^-$$

- **Receivership model**
  - Solvent and liquid firms pay off obligations in full
  - Solvent and illiquid firms receive loan to be repaid at next time period covering total losses
  - Insolvent and illiquid firms pay what they can, rest rolls forward
  - Solvent firms at time t are: $S(t) = \{i \in S(t-1) \mid g_i(V) \geq 0\}$
  - Relative liabilities:

$$a_{ij}^R(t,V) = \begin{cases} 0 & \text{if } i \in S(t), j \neq 0 \\ 1 & \text{if } i \in S(t), j = 0 \\ a_{ij}(t,V) & \text{if } i \notin S(t) \end{cases}$$

  - NOT a nonspeculative system
- **Auction model** can also be constructed (Capponi & Chen (2015))

Discrete-Time Network Model: Receivership Example



- Without loans: $V = (1, -100, -100)^\mathsf{T}$
- With loans: Consider $g_i(V) = V_i + 10$. Two solutions exist:
  - $V = (1, -100, -99)^\mathsf{T}$ with no loans given
  - $V = (2, -1, 0)^\mathsf{T}$ with loan of 1 to firm 1

Dynamic Eisenberg & Noe Financial Contagion Model:

Continuous Time

Continuous-Time Network Model: $\Delta t \to 0$

$$V(t) = V(t-\Delta t) + \dot{c}(t)\Delta t - A(t,V)^{\mathsf{T}} V(t)^{-} + A(t-\Delta t, V)^{\mathsf{T}} V(t-\Delta t)^{-}$$

- $\dot{c}(t)$ is velocity of assets minus liabilities
- Limit provides differential equation:

$$dV(t) = dc(t) - d\left[A(t,V)^{\mathsf{T}} V(t)^{-}\right]$$

- Relative liabilities: $\pi_{ij}(t)$ solves the ODE:

$$V_i(t)^{-} \frac{d\pi_{ij}}{dt}(t) + \left(\sum_{k=0}^{n} L_{ik}(t)\right) \pi_{ij}(t) = L_{ij}(t)$$

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- Existence and uniqueness for nonspeculative system with strictly nonspeculative societal node

Continuous-Time Network Model: Differential Equation

- If $c(t)$ is deterministic:

$$dV(t) = \left(I - \left[A(t,V)^\mathsf{T} - \langle J_x A(t,V), V(t)^- \rangle\right] \operatorname{diag}(V(t) < 0)\right)^{-1} \times \left[dc(t) - \dot{A}(t,V)^\mathsf{T} V(t)^- dt\right]$$

$$\left(\dot{a}_{ij}(t,V)dt + \nabla a_{ij}(t,V)^\mathsf{T} dV(t)\right) V_i(t) + a_{ij}(t,V) dV_i(t) = -L_{ij}(t) + \pi_{ij}(t)\left[\bar{p}_i(t) - V_i(t)^-\right]^+$$

- If $c(t)$ is an Ito process, include appropriate quadratic variation term and $V(t)$ is an Ito process

Continuous-Time Network Model: Eisenberg & Noe

- $L_{ij}(t) \equiv L_{ij}$ is constant over time
- $a_{ij}(t,V) \equiv L_{ij} / \sum_{k=0}^{n} L_{ik} = \pi_{ij}$ is constant over time and wealths
- Wealth differential equation:

$$dV(t) = (I - A^{\mathsf{T}} \operatorname{diag}(V(t) < 0))^{-1} dc(t)$$

- If $c(0) + \int_0^1 dc(t) = x + \sum_{j=0}^{n} (L_{j\cdot} - L_{\cdot j})$ and $V_0 = c(0)$ then V_1 is the Eisenberg & Noe clearing solution

Continuous-Time Network Model: Eisenberg & Noe

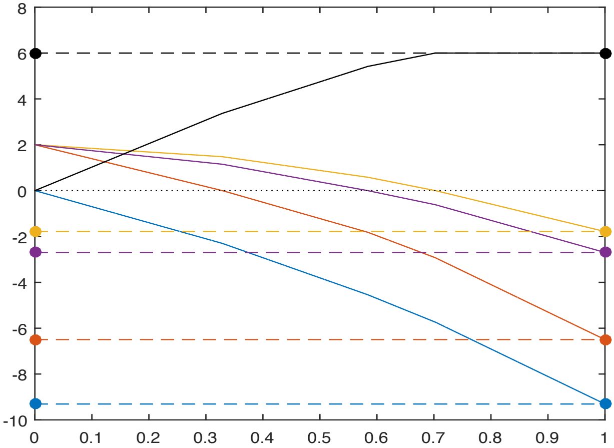


Figure: 4 bank network with $c_0 = x$ and $dc(t) = \sum_{j=0}^{n} (L_{j\cdot} - L_{\cdot j})\, dt$

Continuous-Time Network Model: Eisenberg & Noe

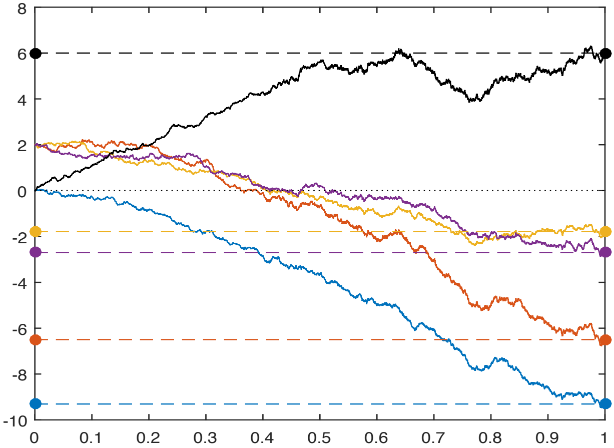
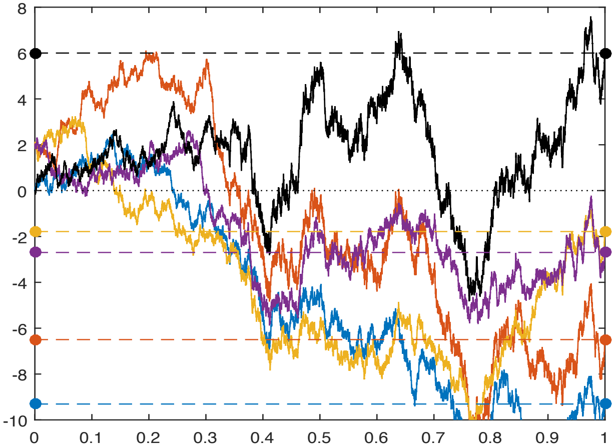


Figure: 4 bank network with $c_0 = x$ and $dc(t)$ Brownian bridge

Continuous-Time Network Model: Eisenberg & Noe



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Thank You!

- EISENBERG, NOE (2001): Systemic risk in financial systems
- BANERJEE, BERNSTEIN, FEINSTEIN (WORKING PAPER): Time dynamic Eisenberg & Noe financial network models