Heterogeneous Preferences and General Equilibrium in Financial Markets

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- In this paper: study the effects of these differences on financial markets. Does the number of types affect outcomes?
- Characterize equilibrium as system of uncoupled ode's $\Rightarrow$ Dimension of state space does not depend on number of types.
- Results:
  - Relationship between variance in EIS and equity risk premium/risk-free rate puzzles.
  - Number of types is important.
  - Returns are predictable as function of dividends.
  - Excess volatility and volatility smile as a low dividend corresponds to high volatility and vice-versa.

References

- How many preference types? Dumas [1989], Bhamra and Uppal [2014], Chabakauri [2015], and Gârleanu and Panageas [2015] focus on two agents. Question remains: Do results generalize quantitatively to many agents?
- Dynamics vs. asymptotics? This paper most closely resembles Cvitanić, Jouini, Malamud, and Napp [2011] and Chabakauri [2015], but I add the explicit study of the effects of changes in the distribution of preferences on short-run dynamics for an arbitrary number of types.
- Asset pricing "puzzles" exist in this model, namely the risk-free rate puzzle (Weil [1989]), the equity risk premium puzzle (Mehra and Prescott [1985]), and the volatility smile (Fouque, Papanicolaou, Sircar, and Sølna [2011]), as well as leverage cycles (Geanakoplos [2010]) and returns predictability (Campbell and Shiller [1988a,b], Mankiw [1981].

Set Up

Details.

- $i \in \{1, 2, ..., N\}$ agents who are price takers
- CRRA instantaneous utility functions, heterogeneous in RRA parameter γ_i

$$U_i(c_{it}) = \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} \qquad \forall i \in \{1, 2, ..., N\}$$

- Assume $\gamma \in [1, \overline{\gamma})$ for exposition.
- Agents preference parameter, γ_i, and their initial wealth, x_i, are drawn from a joint distribution

$$(\gamma_i, x_i) \sim f(\gamma, x)$$

Finanacial Markets

- Risk is driven by a single Brownian motion W_t.
- Trade risky shares, in N Lucas trees paying dividends, $D(t)$, which follows a geometric Brownian motion:

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dW_t$$

- Price of risky and risk free shares, S_t and S_t^0, determined in equilibrium:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$
$$\frac{dS_t^0}{S_t^0} = r_t dt$$

Budget Constraints

- By standard derivations (Merton [1969] or Karatzas and Shreve [1998]) we can arrive at an SDE describing the evolution of agents' wealth X_{it}:

$$dX_{it} = \left[X_{it} \left(r_t + \pi_{it} \left(\mu_t + \frac{D_t}{S_t} - r_t \right) \right) - c_{it} \right] dt + \pi_{it} X_{it} \sigma_t dW_t$$

- Assume non-negative wealth:

$$X_{it} \geq 0, \quad \forall\, t \in [0, \infty) \quad \text{a.s.}$$

- Markets clear:

$$\frac{1}{N} \sum_i c_{it} = D_t \ , \ \frac{1}{N} \sum_i (1 - \pi_{it}) X_{it} = 0 \ , \ \frac{1}{N} \sum_i \pi_{it} X_{it} = S_t$$

Solution: Static Problem

- Following the martingale method of Harrison and Pliska [1981], Karatzas, Lehoczky, and Shreve [1987], define the stochastic discount factor (SDF) H_t as

$$\frac{dH_t}{H_t} = -r_t dt - \theta_t dW_t \qquad where \qquad \theta_t = \frac{\mu_t + \frac{D_t}{S_t} - r_t}{\sigma_t}$$

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- FOC implies:

$$c_{it} = \left(\Lambda_i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}$$

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- Using market clearing gives consumption weights:

$$c_{it} = \omega_{it} D_t \quad \text{where} \;\; \omega_{it} = \frac{N \left(\Lambda_i e^{\rho t} H_t\right)^{\frac{-1}{\gamma_i}}}{\sum_j \left(\Lambda_j e^{\rho t} H_t\right)^{\frac{-1}{\gamma_j}}}$$

Solution: SDF

Proposition

The interest rate and market price of risk are fully determined by the sufficient statistics $\xi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i}$ *and* $\phi_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\omega_{it}}{\gamma_i^2}$ *such that*

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + \phi_t}{\xi_t^3}\sigma_D^2$$

$$\theta_t = \frac{\sigma_D}{\xi_t}$$

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r_t can be rewritten as the rate that would prevail under a single agent with time varying preferences and an extra term

$$r_t = \rho + \frac{\mu_D}{\xi_t} - \frac{1}{2}\frac{\xi_t + 1}{\xi_t^2}\sigma_D^2 - \frac{1}{2}\frac{1}{\xi_t}\left(\frac{\phi_t}{\xi_t^2} - 1\right)\sigma_D^2$$

Solution: CRRA Representative

- Can we match the instantaneous RFR and MPoR with a representative CRRA agent? A Particular Case
- Define γ_{rt} and $\gamma_{\theta t}$ such that

$$r_t = \rho + \gamma_{rt}\mu_D - \gamma_{rt}(1 + \gamma_{rt})\frac{\sigma_D^2}{2}$$
$$\theta_t = \sigma_D \gamma_{\theta t}$$

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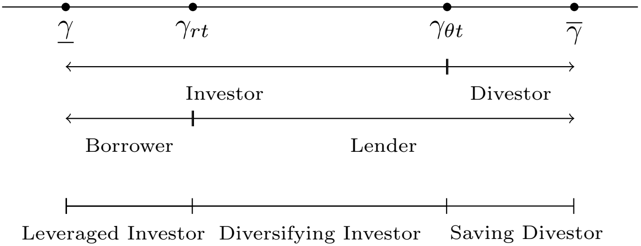
- It can be shown that $\gamma_{rt} < \gamma_{\theta t}$.



Solution: Consumption Weight Dynamics

- Consumption weights follow Ito processes:

$$\frac{d\omega_{it}}{\omega_{it}} = \mu_{it}^{\omega} dt + \sigma_{it}^{\omega} dW_t$$

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$$\sigma_{it}^{\omega} = \theta_t \left(\frac{1}{\gamma_i} - \xi_t \right)$$

- Some agents are buying low and selling high, while others are not.
- Can find consumption weights as functions of the dividend. $\omega_{it} = f_i(D_t)$, such that $f_i(\cdot)$ is an implicit function

$$\frac{1}{N} \sum_j \lambda_{ji}^{\frac{-1}{\gamma_j}} f_i(z)^{\frac{\gamma_i}{\gamma_j}} z^{\frac{\gamma_j - \gamma_i}{\gamma_j}} = 1 \text{ where } \lambda_{ji} = \frac{\Lambda_j}{\Lambda_i}$$

Solution: Wealth/Consumption Ratios

Wealth/consumption ratios $V_i(D) = X_{it}/c_{it}$ satisfy a system of uncoupled ODE's:

$$\begin{aligned} 0 = &1 + \frac{\sigma_D^2 D_t^2}{2} V_i''(D_t) + \left[\frac{1-\gamma_i}{\gamma_i}\theta_t \sigma_D + \mu_D\right] D_t V_i'(D_t) \\ &+ \left[(1-\gamma_i) r_t - \rho + \frac{1-\gamma_i}{2\gamma_i}\theta_t^2\right] \frac{V_i(D_t)}{\gamma_i} \end{aligned}$$

with appropriate boundary conditions. Portfolios are given by:

$$\pi_{it} = \frac{1}{\gamma_i \sigma_t}\left(\gamma_i \sigma_D D_t \frac{V_i'(D_t)}{V_i(D_t)} + \theta_t\right)$$

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- Portfolios exhibit myopic and hedging demand, where hedging demand is always positive.

Solution: Volatility and P/D Ratio

Volatility is given by:

$$\sigma_t = \sigma_D \left(1 + D_t \frac{\mathcal{S}'_t(D_t)}{\mathcal{S}_t(D_t)}\right)$$

where the price/dividend ratio $\mathcal{S}_t(D_t)$ satisfies

$$\mathcal{S}_t(D_t) = \frac{1}{N} \sum_i V_i(D_t)\omega_{it}$$

- Returns determined by $1/\mathcal{S}_t(D_t)$, which is negatively correlated with $W_t \Rightarrow$ Campbell and Shiller [1988a,b], Mankiw [1981].

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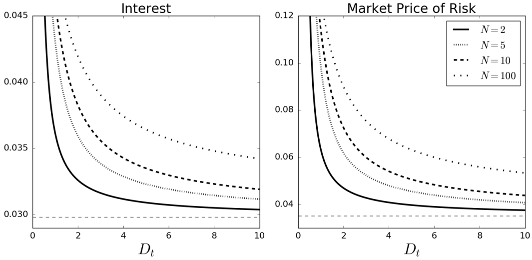
- Returns determined by $1/\mathcal{S}_t(D_t)$, which is negatively correlated with $W_t \Rightarrow$ Campbell and Shiller [1988a,b], Mankiw [1981].
- Model exhibits excess volatility.

Numerical Solution

- Many possibilities, here focus on the effect of number of types.
  - Underlying research question: Is two types sufficient?
  - If so, this additional machinery unnecessary!
- Fix $\gamma_i \sim Uni(1.5, 10.0)$ and change the number of evenly spaced types.
- Fix $\mu_D = 0.01$, $\sigma_D = 0.032$, $\rho = 0.01$. (Chosen to match Chabakauri [2015].)
- Results: Number of types affects level and slope, but not direction of effects.

Numerical Solution: Number of Types

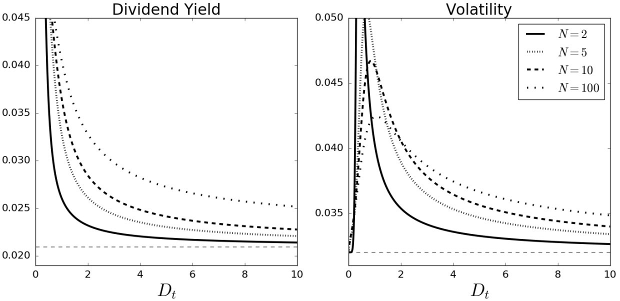
Interest Rate and MPoR



- Changing the number of agents changes financial variables.
- More types generates higher interest rate and MPoR.

Numerical Solution: Number of Types

Volatility and Asset Prices

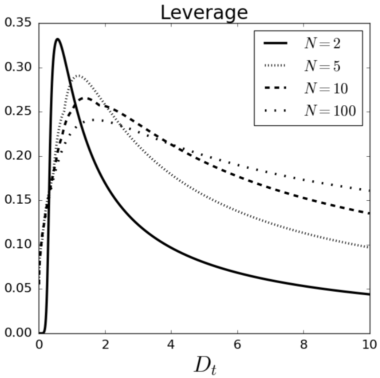


- Increased volatility and negative correlation $\Rightarrow$ volatility smile.
- Negative correlation D/S and D as in Campbell and Shiller [1988a,b] $\Rightarrow$ predictable stock prices.
- Predictability generated by comovement between SDF and consumption as in Mankiw [1981].

Numerical Solution: Number of Types

Leverage

- Fall in D implies a rise in leverage $\Rightarrow$ counter-cyclical leverage cycles.
- Opposite of that assumed/produced in the literature on beliefs generated cycles (Geanakoplos [2010]).
- Complete market allows agents to leverage up in order to smooth consumption.



Conclusion

- Given we are interested in heterogeneous preferences, we should consider the modeling choice of how many types.
- Heterogeneous preferences generate dynamics that match real world data: volatility smile, falling interest rates, predictability of returns, leverage cycles.
- Can partially explain several asset pricing puzzles (risk-free rate puzzle, equity risk premium puzzle, predictability of stock returns).
- Second moment of distribution of preferences matter for RFR and ERP puzzles!
- Looking forward, the introduction of portfolio constraints may provide even better results, in particular for term structure and direction of leverage cycles.

Thanks!

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Additional Materials

The control problem is time inconsistent and non-markovian! Can apply the martingale method (or Girsonov theory) to transform dynamic to static.
Define the stochastic discount factor as

$$H_0(t) = exp\left(-\int_0^t r(u)du - \int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta(u)^2 du\right)$$

where

$$\theta(t) = \frac{\mu_s(t) + \frac{D(t)}{S(t)} - r(t)}{\sigma_s(t)}$$

represents the market price of risk. This implies that the stochastic discount factor also follows a diffusion of the form

$$\frac{dH_0(t)}{H_0(t)} = -r(t)dt - \theta(t)dW(t)$$

Using the stochastic discount factor, we can rewrite each agent's dynamic problem as a static one beginning at time $t = 0$

$$\max_{\{c^i(u)\}_{u=0}^{\infty}} \quad \mathbb{E} \int_0^{\infty} e^{-\rho u} \frac{c^i(u)^{1-\gamma_i} - 1}{1-\gamma_i} du$$

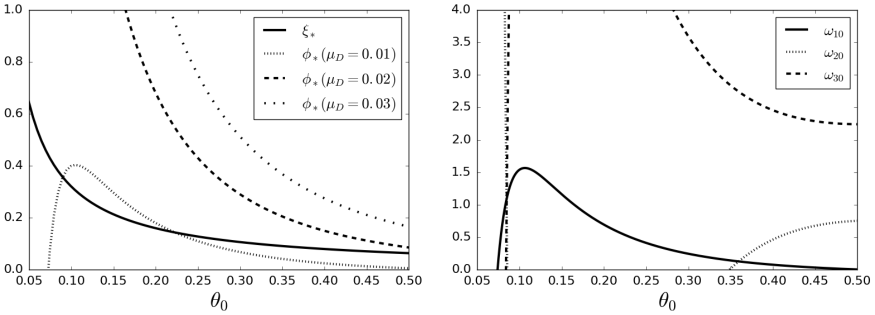
$$\text{s.t.} \quad \mathbb{E} \int_0^{\infty} H_0(u) c^i(u) du \leq x_i$$

First order condition by calculus of variations:

$$c^i(t) = \left(\Lambda_i e^{\rho t} H_0(t) \right)^{\frac{-1}{\gamma_i}}$$

Return

- Fix $r_0 = 0.03$, $\sigma_D = 0.032$, $\rho = 0.02$, and plot the values of ξ_t and ϕ_t which give θ for different values of μ_D. **(plot a)**

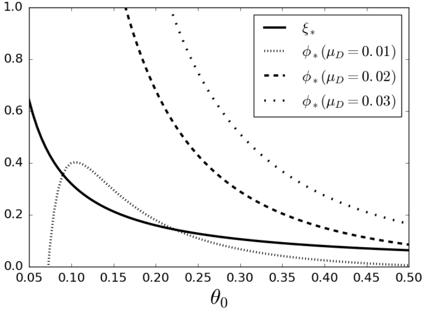


(a) (b)

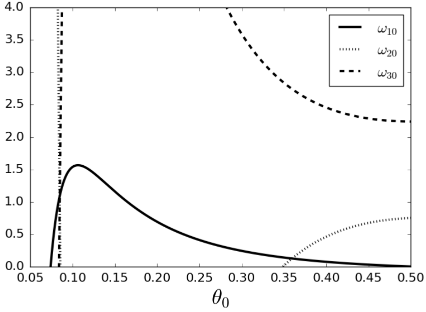
Solution: Matching Equity Risk Premium

Return

- Fix $r_0 = 0.03$, $\sigma_D = 0.032$, $\rho = 0.02$, and plot the values of ξ_t and ϕ_t which give θ for different values of μ_D. **(plot a)**
- Consider three agents with preference parameters fixed to $(\gamma_1, \gamma_2, \gamma_3) = (1.1, 10, 18)$. Could these agents produce observed ERP? **(plot b)**



(c)



(d)

Proposition

Assuming consumption weights also follow a geometric Brownian motion such that

$$\frac{d\omega^i(t)}{\omega^i(t)} = \mu_{\omega^i}(t)dt + \sigma_{\omega^i}(t)dW(t)$$

then $\mu_{\omega^i}(t)$ and $\sigma_{\omega^i}(t)$ are given by:

$$\mu_{\omega^i}(s) = (r(t) - \rho)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \theta(t)^2\left[\left(\frac{1}{\gamma_i^2} - \phi(t)\right) - 2\xi(t)\left(\frac{1}{\gamma_i} - \xi(t)\right) + \left(\frac{1}{\gamma_i} - \xi(t)\right)\right]$$

$$\sigma_{\omega^i}(t) = \theta(t)\left(\frac{1}{\gamma_i} - \xi(t)\right)$$

Convergence: Consumption Weights

Recall the definition of $\omega^i(t) = \omega(\gamma^i, x^i, t)$ and consider the limit in N

$$\frac{N\left(\Lambda_i e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_i}}}{\sum_{j=1}^{N}\left(\Lambda_j e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma_j}}} \underset{N\to\infty}{\to} \frac{\left(\Lambda(\gamma,x) e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma}}}{\int \left(\Lambda(\gamma,x) e^{\rho t} H_0(t)\right)^{\frac{-1}{\gamma}} dF(\gamma,x)}$$

by the law of large numbers, which implies $\omega(\gamma^i, x^i, t) \underset{N\to\infty}{\to} \omega(\gamma, x, t)$. A similar result holds for $\xi(t)$ and $\phi(t)$, as well as for financial variables (e.g. $r(t)$, $\theta(t)$, etc.).

Convergence: Interpretation I

In the continuous types case, $\omega(\gamma, x, t)$ is the Radon-Nikodym derivative of the initial distribution $F(\gamma, x)$ with respect to another, stochastic distribution:

$$
\begin{aligned}
1 &= \int \omega(\gamma, x, t) dF(\gamma, x) \\
&= \int \frac{dG(\gamma, x, t)}{dF(\gamma, x)} dF(\gamma, x) \\
&= \int dG(\gamma, x, t)
\end{aligned}
$$

Then $\omega(\gamma, x, t)$ represents the dynamics of the intinite dimensional, Banach valued random process $G(\gamma, x, t)$.
Is this the optimal transport? Can this be thought of as the solution to the Munge problem?

Convergence: Approximation

It can be shown that if one is attempting to match the continuous types approximation with a histogram (which is equivalent to discrete types), the best one can do is

$$G(A,0) = \int_A \omega(\gamma, x, 0) f(\gamma, x) d\gamma dx = \int_A \frac{1}{(\overline{\gamma} - \underline{\gamma})(\overline{x} - \underline{x})} d\gamma dx$$

That is, one could only match an initial condition where the product $\omega(\gamma, x, 0) f(\gamma, x)$ is a uniform distribution.

The continuous types model allows for a greater amount of freedom with less computational cost.

Preference Levels

The preference levels which clear the market are given by

$$\gamma_r(t) = \frac{\mu_D}{\sigma_D^2} - \frac{1}{2} - \sqrt{\left(\frac{\mu_D}{\sigma_D^2}\right)^2 - \frac{\mu_D}{\sigma_D^2}\left(1 + \frac{2}{\xi(t)}\right) + \frac{\xi(t) + \phi(t)}{\xi(t)^3} + \frac{1}{4}}$$

$$\gamma_\theta(t) = \frac{1}{\xi(t)}$$